A Constructive Approach to the Non-Riemannian Features of Dislocation and Spin Distributions in Terms of Finsler's Geometry and a Possible Extension to the Space-Time Formalism

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[Received April 1967]

INTRODUCTION

Among many things which we have learned in connexion with our endeavour to construct the mathematical theory of plastic manifolds, we are particularly impressed by the fact that one and the same physical object permits a number of apparently different representations even using geometrical terminology. The difference originates sometimes from the different aims because of which the investigators are interested in different phases of the same subject. However, even with the same aim, various representations have been offered according to the different degrees of penetration and the different degrees of unification. In any case, a geometrical language with which we describe the physical phenomena should not be assumed to be the final one. As we have so many sorts of geometrical systems, which differ from one another in regard to the standpoints and the degrees of accuracy, what appears to be excluded by one of them can be introduced from the standpoint of another. This may appear to be very inconvenient at first sight. However, the truth lies in the very fact that our human form of recognition of anything is hardly final and the geometry is no exception. Nor can the physical recognition escape constant modification.

The present exposition of the Finslerian aspects of plasticity and magnetism is an illustration of the possibility of different approaches to the same problem. The non-Riemannian treatment of problems of continuous distribution of dislocations has become rather conventional. One might assume that the Cosserat continua are the non-Riemannian spaces equipped with the linear connexion with a torsional structure. But different expressions should also be noted. We have already a Theory of Yielding in which a different approach is observed.

In what follows we shall emphasize constructional meanings of the conventional concepts which are compared to the plastic, magnetic and other kinds of anomalous objects classified from the conventional physical point of view. Since we construct, we need to start with more microscopic structures. Hence we need a more microscopic geometry than hitherto. So far, the language of linear affine connexion has been fashionable. However, the geometry has provided us something more with which we can penetrate more microscopically. A thorough investigation along such lines is being made in another Division of these Memoirs (Division E). The geometry initiated by P. Finsler1) allows us the first step of such a penetration. Using it, we first decompose a part if not all, of the anomalies into the distribution of Finsler’s elements of support and then we construct the conventional torsion and curvature fields which have so far been regarded

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** Reference [1]; see also [2] and [3].
to be the most appropriate concepts to represent the dislocation and incompatibility distributions etc. In other words, the following construction affords a more natural introduction of such a representation and a feasibility of an extension and unification. As a geometry itself, this is obvious. As a physics, it is more constructive.

In fact, it is only a single step to unify the description of the spin distribution in ferromagnetism with the dislocation distribution in plasticity.

The first section is a summary of Finsler's geometry from the standpoint of the subsequent applications. The torsional structure is intentionally restricted by Cartan's postulates so that the construction of the ordinary torsion by osculation in the second section will appear more impressive. The third section is an illustration of a unification of apparently different physics, the dislocation and spin, and the meaning of their distinction.

In the fourth section we shall study how the curvature features are responsible for different anomalous or physical aspects and how they are expressed when the osculation has been performed. It is natural that an interpretation of the meaning of the principle of general relativity is given in this connexion from the constructional standpoint.

We do not attempt to develop all the possible formulae that can be reached in this way. We illustrate how a more macroscopic geometrical summary is constructed by approximation or degeneration from a more microscopic one. Similarly, we would emphasize that a more microscopic classification can be included in an apparently different form in a more macroscopic formulation.

1. General Standpoint of Finsler's Geometry

1.1. Physical phenomena depending on direction. A physical body is generally full of all sorts of anomalies and misfits so that its elements exert constraints on one another in order to keep them continuously fitted into space. Any material line or chain of material points in it is also under the constraint of its surroundings. Hence it has, in the constrained state, a different form and a different length from those which it assumes when it is cut off from the surroundings and freely unstrained.

If the nature of the constraint is so relatively simple that their distribution in the neighbourhood of an arbitrary point is fully described by the unstraining of small elements drawn from it to its neighbouring points, then the aspects of the local constraint configuration are given as functions of the set

\[(x^i, dx^i) \tag{1.1}\]

where \(i = 1, 2, 3\). Among the various possibilities, one can confine oneself to the simpler kind in which the length of the element is irrelevant, the direction alone being sufficient to fix the physics of the constraints. In that case, (1.1) is sufficiently represented by

\[(x^i, \dot{x}^i) \text{ or simply } (x, \dot{x}), \tag{1.1'}\]

where

\[\dot{x}^i = \frac{dx^i}{dt}\]

and \(t\) is an arbitrary parameter. It is then necessary that the physical quantities depend on the ratios

\[dx^1 : dx^2 : dx^3 = \dot{x}^1 : \dot{x}^2 : \dot{x}^3 \tag{1.2}\]

so that they are homogeneous functions of \(\dot{x}^i\).

Such a set (1.1) or (1.1') is called the element of support in Finsler's and Cartan's sense ([1] and [2]).

Therefore, to assume an element of support is to shield off, from the surroundings a material line-element at \(x^i\) having the direction (1.2). Whatever the nature of the constraint from which it is so shielded off, the length it assumes at the shielded condition depends on (1.1') so that it is given by

\[ds = F(x^i, \dot{x}^i) dt \tag{1.3}\]

where the length \(ds\) and the differential of the parameter \(t\) are proportional within the neighbourhood where the physics is entirely represented by \(F(x^i, \dot{x}^i)\). It is necessary that \(ds\) or the integral

\[s = \int_{t=t_1}^{t=t_2} F(x^i, \dot{x}^i) dt \tag{1.4}\]

has an invariant meaning in spite of possible transformations of the parameter \(t\). Some restrictions are imposed by it. They are summarized into the following conditions, most of which are
self-evident.\footnote{See, for instance, H. Rund \cite{3}. We adopt mostly Rund's notations.}

**Condition I.** The function \( F(x^i, \dot{x}^i) \) is positively homogeneous of degree 1 in \( \dot{x}^i \).

Evidently Condition I entails
\[
\dot{x}^i \partial_{\dot{x}^i} F = F
\]  
(1.5)

where
\[
\partial_{\dot{x}^i} = \frac{\partial}{\partial \dot{x}^i}.
\]

**Condition II.** The function \( F(x^i, \dot{x}^i) \) is positive,
\[
F(x^i, \dot{x}^i) > 0
\]  
(1.6)

with
\[
\sum_i (\dot{x}^i)^2 = 0,
\]
i.e. if not all \( \dot{x}^i \) vanish simultaneously.

**Condition III.** Only positive values are assumed by the quadratic form \( g_{ij}(\xi^i \xi^j) \), where
\[
g_{ij} = \frac{1}{2} \partial_{\dot{x}^i} \partial_{\dot{x}^j} F^2,
\]  
(1.7)

unless the \( \xi^i \) vanish simultaneously and for all elements of support which appears as the argument in the coefficients \( g_{ij} \).

The following identities are obtained
\[
\dot{x}^i \dot{x}^j \partial_{\dot{x}^i} \partial_{\dot{x}^j} F = 0,
\]  
(1.8)

\[
\frac{1}{2} \left( \partial_{\dot{x}^i} \partial_{\dot{x}^j} F^2 (x, \dot{x}) \right) \dot{x}^i \dot{x}^j = F^2
\]  
(1.9)

or
\[
F^2 (x, \dot{x}) = g_{ij}(x, \dot{x}) \dot{x}^i \dot{x}^j.
\]

1.2. Tangent Euclidean space. The foregoing specification leads to the concept of a higher order space which was first studied by P. Finsler \cite{1}. In order to develop the analysis of all sorts of geometric objects associated with Finsler's space, it is first necessary to know a more accurate specification of the released element in Euclidean space.

From our point of view that the element of support is a material element, it is also meaningful to consider another vector \( X^i \) attached to the same material element (Fig. 1). Both \( X^i \) and \( \dot{x}^i dt \) being immersed in Euclidean space, all sorts of Euclidean specifications are defined between them. This affords the concept of the tangent space attached to the element of support.

![Fig. 1](image)

In the first place, the magnitude of the vector \( X^i \) should be defined. This is immediately given if \( X^i \) has the same direction as \( \dot{x}^i dt \) so that
\[
X^i = a \dot{x}^i dt
\]  
(1.10)

where \( a \) is a scalar. Obviously the length of \( X^i \) should be \( a \) times the length \( ds \) of \( \dot{x}^i dt \). This is in fact the case since
\[
F(x^i, X^i) = a F(x^i, \dot{x}^i) dt = a ds.
\]

From (1.9), the squared length can be written as
\[
F^2(x^i, X^i) = a^2 g_{ij}(x, \dot{x}) \dot{x}^i \dot{x}^j dt^2.
\]

But this is for a particular direction corresponding to (1.10) in Euclidean space. For an arbitrary direction of \( X^i \), the corresponding quantity should be
\[
X^a \overset{def}{=} g_{ij}(x, \dot{x}) X^i X^j.
\]  
(1.11)

Therefore, we have the following postulate.

A. A symmetric tensor \( g_{ij}(x, \dot{x}) \) is given such that the square of the magnitude of an arbitrary vector \( X^i \) is defined by the quadratic form
\[
X^a \overset{def}{=} g_{ij}(x, \dot{x}) X^i X^j.
\]

Similarly for the resultant vector
\[
X^i + \dot{x}^i dt
\]
so that the square of its magnitude is
\[(X + \dot{x}dt)^2 = g_{ij}(X^i + \dot{x}^i dt)(X^j + \dot{x}^j dt)\]
\[= X^a + \dot{x}^a + 2g_{ij}(x, \dot{x})X^i \dot{x}^j dt\]
\[= X^a + \dot{x}^a + X^i dt \partial_x F^i(x, \dot{x}).\]

If
\[X^i \partial_x F^i(x, \dot{x}) = 2g_{ij}(x, \dot{x})X^j \dot{x} = 0,\]  
then the square of the magnitude of the resultant vector \(X + \dot{x}dt\) has the structure of Pythagoras' theorem, \((X + \dot{x} dt)^2 = X^2 + (\dot{x} dt)^2\). The vector \(X^i\) satisfying (1.12) is also called \textit{transversal}.

As Cartan put it, one can relate this feature as

\textbf{B. Any vector which is transversal to its element of support is perpendicular to that element.}

It should be remarked that the metric tensor \(g_{ij}(x, \dot{x})\) changes when the direction of shielding, i.e., that of the element of support, changes. In Fig. 2, the metric configuration produced for the shielding along \(AB\) is generally different from that produced for the shielding along \(CD\). In dealing with two such manners of shielding, we are talking about two tangent Euclidean spaces. The mutual relations connecting them will be studied next.

\textbf{1.3. Euclidean connexion.} The foregoing postulates A and B are but an immediate consequence associated with the concept of element of support. They are responsible only for vectors attached to an element of support. We need to supplement them by fixing what is required by the metric condition on the manner of comparison of physical fields attached to different elements of support.

Element by element, the releasing of the material manifold effects on the appearance of the distribution of the field, in a specific manner depending on the location of the centre and the direction of the shielding of the element of support. It is required to know how the appearance at an element is connected with that at another. This is to compare the conditions at two such tangent spaces as have been defined in the preceding sections. Therefore, it is rightly called a \textit{Euclidean connexion}.

We may start with the basic

\textbf{Requirement I.} There should be an analytical quantity which expresses the true change of an arbitrary field of vector \(X^i\) between two neighbouring elements of support. The vanishing of that quantity should mean that there is no true difference of the field quantity \(X^i\) between the two configurations of releasing the material elements from the constraints.

Obviously, the analytical expression should be linear with respect to the small change of the labels from \((x^i, \dot{x}^i)\) to \((x^i + dx^i, \dot{x}^i + d\dot{x}^i)\). Moreover, the field difference disappears if the field itself vanishes. Hence the former can also be linear in regard to the latter, up to the first approximation at least. Thus we have the basic concept which is defined as follows.

\textbf{Definition.} Under an infinitesimal change of the element of support, the vector \(X^i\) is varied through
\[DX^i = dX^i + C_{ik}^h(x, \dot{x})X^h d\dot{x}^k + \Gamma_{ik}^h(x, \dot{x})X^k dx^h\]  
(1.13)

which is called the covariant (or absolute) differential where \(C_{ik}^h\) and \(\Gamma_{ik}^h\) are also functions of the element of support.

The condition that there is no true difference of the field quantity \(X^i\) between the two neighbouring elements is obviously given by
\[DX^i = 0,\]
(1.14)
which indicates that the vector field is \textit{parallelly displaced} between these elements in the released condition. This is materialized by the

\textbf{Requirement II.} The covariant derivative \(DX^i\), or the parameters involved in it, should be chosen in such a manner that, if it vanishes, or
\[dX^i = -C_{ik}^hX^h d\dot{x}^k - \Gamma_{ik}^hX^k dx^h,\]  
(1.14-1)
then the vector \(X^i\) is transported parallelly from \((x^i, \dot{a}),\)
\( \dot{x}^i \) to \( (x^i + \dot{x}^i, \dot{x}^i + \dot{\dot{x}}^i) \) in the released state and vice versa.

It is included in the foregoing that the length of an arbitrary vector \( X^i \) such as has been defined by (1.11), remains invariant under the parallel displacement. From the covariant differentiation it follows that

\[ DX^i = (Dg_{ij}) X^j X^i = 0 \quad (1.15) \]

whence

\[ Dg_{ij} = 0. \quad (1.16) \]

The formula (1.16) indicates that the metric tensor is insensitive of the covariant differentiation. From this it follows that

\[ \partial_k g_{ij} = \Gamma_{ik}^k + \Gamma_{jk}^i \quad (1.16-1) \]

and

\[ \partial_k g_{ij} = C_{ik}^k + C_{jk}^i \quad (1.16-2) \]

where

\[ \Gamma_{ik}^j = g_{jk} \Gamma_{ik}^k, \quad C_{ik}^j = g_{jk} C_{ik}^k \]

and

\[ \partial_k = \frac{\partial}{\partial x^k}. \]

Assuming a certain symmetry condition, we may state that

\[ C_{ijk} = \frac{1}{2} \partial_j c_{ik} + \frac{1}{4} \partial_i \partial_j \partial_k \Gamma^a (x, \dot{x}). \quad (1.17) \]

We have

\[ \dot{x}^k C_{ijk} = \dot{x}^i C_{ijk} = \dot{x}^a C_{ijk} = 0, \quad (1.18) \]

i.e. \( C_{ijk} \) is positively homogeneous of degree -1 in regard to \( \dot{x}^i \), which leads to the following postulate.

C. Any vector with the same direction as the element of support has its covariant differential identically annulled when its elements of support are rotated about its centre and the contravariant components are unchanged.

If the element of support is displaced parallely to itself, then

\[ DI^i = 0 \]

or

\[ \dot{x}^i = \dot{x}^i \left( \frac{dP^i}{dP} \right) - \Gamma^i_{jk} \dot{x}^j \dot{x}^k \]

so that, on substituting this in (1.13),

\[ DX^i = dX^i + \Gamma^i_{jk} X^j \dot{x}^k \quad (1.13-1) \]

where

\[ l^i = \frac{\dot{x}^i}{\Gamma^i}, \quad (1.19) \]

\[ \Gamma^i_{jk} = \Gamma^i_{kj} - C_{kj}^k \Gamma^i_{kj} \quad (1.20) \]

By some calculation we obtain

\[ \Gamma^i_{jk} = \gamma_{(kj)} - C_{(k}\partial_x \gamma_{j)} G^k - C_{k} \partial_x \gamma_{j} G^k = C_{k} \partial_x \gamma_{j} G^k \]

(1.20-1)

where

\[ \gamma_{ij} = g_{jk} \Gamma^k_{ij}, \quad G^k (x, \dot{x}) = \frac{1}{2} \gamma_{ij} (x, \dot{x}) \dot{x}^i \dot{x}^j \quad (1.21) \]

and

\[ \gamma_{ij} = \frac{1}{2} \left( \partial_k g_{ij} (x, \dot{x}) + \partial_i g_{jk} (x, \dot{x}) - \partial_j g_{ik} (x, \dot{x}) \right). \]

Also

\[ \Gamma^i_{kj} = \gamma_{(kj)} + C_{k} \partial_x \gamma_{j} G^k \]

(1.20-1)

It should be remarked that the parameters \( \Gamma^k_{ij} \) or \( \Gamma^i_{kj} \) are symmetric with respect to the indices \( j \) and \( k \).

These propositions A, B, C, and some minor characteristics mentioned in the foregoing are the same as those which have been adopted by Cartan in [2] as the basic postulates to fix the Euclidean connexion unambiguously.

1.4. Torsion. An important physical interpretation is associated with the concept of parallelism as follows.

Consider two displacements \( dx^i \) and \( dx^i \). Let them be represented by segments \( PP_2 \) and \( PP_3 \) respectively in the Euclidean space into which all the elements of support have been released. In the endeavour to construct an infinitesimal parallelogram by parallely transporting \( PP_2 \) from \( P \) to \( P_2 \), and \( PP_3 \) from \( P \) to \( P_3 \), the points \( P_1 \) and \( P_2 \) so transported fall on \( P_3 \) and \( P_3 \) respectively.

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entailing a closure failure $\mathcal{D}_{\mathcal{M}}^P$ (Fig. 3). This is the concept called torsion. The failure is given by

$$\mathcal{Q}^i = [dx^k \omega^i_j]^j = d_x^k \omega^i_j - d_x^k \omega^i_1 \quad (1.23)$$

where $\omega^i_j$ are the values of $\omega^i_j$ respectively on $dx$ and $d_x$, where

$$\omega^i_j = A^i_{kh} D_x^k + \Gamma^i_{kh} d_x^k \quad (1.24)$$

On substituting (1.24) in (1.23) and by some calculation, we have

$$\mathcal{Q}^i = 2 \lambda^i_{kh} d_x^k \quad (1.25)$$

where terms involving $\Gamma^i_{kh}$ have dropped off owing to its symmetric structure in regard to $h$ and $k$.

If $D_x^k = 0$, we have

$$\mathcal{Q}^i = 0,$$

which means the following lemmas.

**Lemma 1.** No closure failure is produced by describing a small cycle by parallel displacement entirely within the restricted region where all the elements of support are parallel to one another (Fig. 4).

**Lemma 2.** In order that a closure failure is entailed, it is necessary to have elements of support of different orientations and to describe a cycle by parallel displacement over them.

2. Osculating Spaces

2-1. An osculating Riemannian space. So far no restriction has been imposed on which direction the element of support should be defined in the material body. Let us now restrict the problem by so fixing the direction of each element. The direction coordinates are then given as functions of the coordinates of the centre of support $x^i$:

$$\dot{x}^i = \xi^i (x). \quad (2.1)$$

The components of the metric tensor become also functions of $x^i$ alone:

$$\gamma_{ij}(x) = g_{ij}(x, \xi(x)) \quad (2.2)$$

to define a Riemannian metric. The manifold so defined is called the osculating Riemannian space.

The word osculating is appropriate since the definition depends still on the material chain along which the released length $s$ is measured.

An osculation is realized if two neighbouring material grains in the material chain are fused together. By the substitution (2.1), the direction coordinates apparently disappear so that a new parallelism can be defined solely from the new metric tensor $\gamma_{ij}(x)$ depending on the coordinates of the element centre $x^i$ alone. As a natural consequence, the Levi-Civita parallelism defined by the new metric tensor $\gamma_{ij}$ is derived from the ordinary parallelism in the Euclidean space into which the fused material chain is released. For any vector field $X^i(x)$ it gives the covariant differentiation

$$D_n X^i = dx^i + \left[ \sum_{k \neq i} \right] X^k \quad (2.3)$$

where

$$\left[ \sum_{k \neq i} \right] X^k = \frac{1}{2} \gamma^{ik} (\partial_{kT} + \partial_{kT} - \partial_{kT}) \quad (2.4)$$

and $\gamma^{ik}$ is the contravariant fundamental tensor corresponding to $\gamma_{ij}$.

2-2. Construction of an osculating non-Riemannian space. On the other hand, it is possible to consider connexions different from that which
appears in the osculating Riemannian manifold of §2.1. It means that two neighbouring elements of support are set and fused together in different manners from the one initially given. All such non-Riemannian possibilities with the metric character are included in the general formulae studied in the foregoing using Finsler's and Cartan's picture. We have simply to select specific functions of \( x^i \) and \( \dot{x}^i \) for \( g_{ij} \) and \( \Gamma^i_{jk} \) and to substitute (2-1) for the \( \dot{x}^i \) in them. We then have, from (1-13),

\[
DX^i = dX^i + \Pi^i_{jk}(x) X^k dx^j
\]

where

\[
\Pi^i_{jk}(x) = C^i_{kh}(x, \xi(x)) \partial_j x^h + \Gamma^i_{jk}(x, \xi(x)) = C^i_{kh}(x, \xi(x)) D_j x^h + \Gamma^i_{jk}(x, \xi(x)),
\]

where

\[
D_j x^h = \partial_j x^h + C^h_{kb} \xi^b \partial_j \xi^k + \Gamma^h_{jk}(x, \xi(x)).
\]

Both \( \Pi^i_{jk} \) and \( \begin{pmatrix} i \\ k \end{pmatrix} \) have the character of linear affine connexion. In other words, they are transformed under the transformation of the coordinates from \( x^i \) to \( \bar{x}^a \) by

\[
\begin{pmatrix} a \\ b \end{pmatrix} = \partial_a x^k \partial_b x^j \partial_j x^a
\]

and

\[
\Pi^a_{bj} = \partial_a x^k \partial_b x^i \partial_i x^a \Pi^i_{kj} - \partial_a x^k \partial_b x^j \partial_i x^a \partial_j x^a.
\]

Hence

\[
T^i_{kj} = \Pi^i_{kj} - \begin{pmatrix} i \\ k \end{pmatrix} = C^i_{kh}(x, \xi) D_j x^h + \Gamma^i_{jk} - \begin{pmatrix} i \\ k \end{pmatrix}
\]

are transformed like a tensor.

Since

\[
Z^i_{kj} = C^i_{kh}(x, \xi) D_j x^h
\]

is a product of the tensors \( C^i_{kh} \) and \( D_j x^h \), it is also a tensor. It follows that

\[
\Gamma^i_{kj} - \begin{pmatrix} i \\ k \end{pmatrix} = T^i_{kj} - Z^i_{kj}
\]

is also a tensor and \( \Gamma^i_{kj} \) have also the character of linear affine connexion.

The linear affine connexion defined by \( \Pi^i_{jk} \) needs to be metric, i.e. the metric tensor \( \gamma_{ij} \) is annulled by the covariant differentiation defined by it since the original covariant differentiation in the manifold, from which this has been derived, has the same character. In other words, the linear connexion we have obtained for the tensor \( \gamma_{ij} \) and parameters \( \Pi^i_{jk} \) is Euclidean or metric.

It is a simple analysis to prove that the tensor part of

\[
\Pi^i_{kj} = \Pi^i_{kj} g_{hi}
\]

i.e. \( T_{ij} \) is antisymmetric with respect to the last two indices:

\[
T_{ikj} = T_{kj i}, \quad T_{kij} = 0,
\]

(2-7)

(see [4]).

The space of linear connexion so introduced may be called an osculating non-Riemannian space.

2.3. A more natural approach to the theory of continuous distributions of dislocation. From the metric condition (2-7) it immediately follows that

\[
T_{ikj} = S_{ikj} + S_{ik} g^{th} g_{th} + S_{ih} g^{ih} g_{ij}
\]

or

\[
T_{ikj} = S_{ikj} + S_{ik} g^{th} + S_{ih} g^{ih},
\]

(2-8.1)

where \( S_{ikj} \) is a tensor which is antisymmetric with respect to the lower indices. Obviously

\[
S_{ikj} = T_{ikj} - Z_{ikj} = C^i_{kh} D_j x^h.
\]

(2-9)

It is simply a recapitulation of two well known propositions to show that, by the parallel transportation of vectors \( PP^i \) (or \( dx^i \)) and \( \bar{PP}^i \) (or \( \bar{dx}^i \)) from \( P \) to \( P^i \) and \( P^i \) respectively, generally an unclosed parallelogram is entailed. This time the closure failure is given by

\[
\Omega^i = 2 S_{ikj} d x^k d x^j.
\]

(2-10)

The same failure should be obtained in the manner of §1-4 where the direction coordinates \( \bar{\xi}^b \) are replaced by \( \xi^b(x) \). Also, from (1-26), by direct substitution, we have the same formula. Therefore, the torsion concept defined in §1-4 for the Finsler space is immediately brought over to the space of linear connexion where the direction parameters are fixed by (2-1). The tensor
\( S_{ij} \) so defined by (2.9) is called the torsion tensor.

The corresponding Finslerian tensor \( C_{ij} \) or \( A_{ij} \) has also been called the torsion tensor in Finsler’s space.

Since much has been done in the study of the dislocation distributions in non-Riemannian terminology, it is no longer necessary to point out that the torsion tensor is an appropriate expression of the density of dislocations. (See, e.g., [5], [6]). A torsion concept is automatically introduced by the Finslerian criterion and reduced by the osculation to be the well known expression of the dislocation density. The torsion can approximately be realized by a continuous distribution of dislocations but not necessarily by a single dislocation line.

It is necessary to refer to Lemmas 1 and 2 of §1.4. In reality, the osculating manifold consists of small perfect crystal elements in different orientation from each other. In each of them the releasing cuts or elements of support can be uniform in direction. Hence, no closure failure is entailed within such a grain. The torsion is, therefore, a geometric object defined over more than one such grain or unit of a mosaic structure in an imperfect crystal.

The foregoing construction affords, therefore, a more natural approach than hitherto to the geometrical theory of plasticity.

2.4. Finslerian character of the ferromagnetic magnetostriction. The osculation is, however, not necessarily required. Generally, the parameters \( \xi^i \) represents additional physical degrees of freedom other than the space-fixed dislocations. They can represent physical phenomena in which the metric properties depend on the direction or some velocity components including angular velocities defined at each material point.

We know at least one of such cases regarding the metric of the ferromagnetic substances in which the direction parameters are needed to describe certain angular momenta.

At each point of a ferromagnetic substance a spontaneous magnetization is caused mainly by the spin of the atoms. The spin has the character of angular momentum quantum-mechanically. Moreover, in the case of ferromagnetism, the magnetic moment density of a magnetic domain has a constant magnitude. If its components are \( \gamma^k \), we have, on normalization,

\[
a_{ij} \gamma^i \gamma^j = 1 \tag{2.11}
\]

where \( a_{ij} \) is an appropriate normalization tensor.

This \( \gamma^i \) is evidently a special case of \( \xi^i \) [defined by (1.19)].

Owing to an interaction between the magnetization and the deformation, a magnetostriction arises so that the material obtains a natural metric depending not only on \( \xi^k \) but also on \( \xi^i \) (\( \gamma^k \)). This is evidently Finslerian, provided the metric conditions considered in §1 are satisfied.

An essay to analyze magnetostriction with this picture has been made by Amari [7]. He considers essentially a series expansion

\[
g_{ij} = \delta_{ij} - 2 \epsilon_{ij} - \frac{1}{2} \gamma_{ijkl} \xi^l \xi^k \cdots
\]

where

\[
g_{ij}(x, l) = g_{ij}(x, -l)
\]

is assumed, taking account of the symmetry of the magnetostriction. The metric is only slightly deviated from the Euclidean condition in which the coordinates \( \xi^i \) (\( i = 1, 2, 3 \)) act as rectangular Cartesian. In other words, the metric

\[
ds^2 = g_{ij} \, d\xi^i \, d\xi^j
\]

is analyzed in comparison with the Euclidean metric

\[
ds^2 = \delta_{ij} \, dx^i \, dx^j
\]

which is the special case for

\[
a_{ij} = 0 \quad \text{and} \quad \gamma_{ijkl} = 0.
\]

By osculation, the Finslerian space can be reduced to a space of linear Euclidean connexion.

A torsion tensor having the same analytical structure as the \( S_{ij} \) of (2.9) is introduced. But it is, of course, due entirely to the magnetostriction. Owing to the continuous distribution of the spin direction, its Burgers vector need not be an integral multiple of a lattice vector. Hence it gives a dislocation cloud.\(^1\) The Bloch wall can be analyzed.\(^2\)

It is obvious that one can also start from the beginning with this reduced picture in terms of the geometry of linear affine or metric connexion where the torsion tensor of the ordinary kind represents some aspects of the spin distribution.

The continuum, in which the spin distribution

\(^1\) See [7], also [8], [9] etc.
\(^2\) See [7].
is so assumed, can be extended to a four-dimensional space-time, if it is preferred, in order apparently to fit the viewpoint with the general relativistic formalism. There was, indeed, some such endeavour (e.g. [10]). Of course, this can be arrived at by the reduction of a four-dimensional Finslerian picture by the osculation in the foregoing manner. We shall return to this problem later (in §4).

3. Coexistence of Dislocations and Spin Domains

The two foregoing interpretations may appear to exclude one another. If the standpoint of §2·3 is assumed, the spin degrees of freedom seem to escape. If that of §2·4 is assumed, the ordinary dislocations seem to escape. In order to include both, we need to extend the space concept a little further. This has been done by relaxing Cartan’s restriction on the parallelism to some measure so that the ordinary kind of torsion co-exists from the beginning with Finsler’s $C_{i,jk}$ and $\Gamma^i_{j,k}$.

Here we shall present a somewhat different exposition starting from the space concept with all the restrictions adopted in §1.

3·1. A six-dimensional Finsler space. To include additional degrees of freedom is to elevate the dimensionality of the space. Let us assume in addition to the three ordinary space dimensions responsible for the three crystallographic axes, for example, three more degrees of freedom. Correspondingly the metric obtains a composite six-dimensional structure

$$d\bar{s}^2 = ds^2 + 2d\bar{s} \cdot d\bar{s} + d\bar{s}^2,$$ (3·1)

where

$$d\bar{s}^2 = g_{ij} \bar{d}x^i \bar{d}x^j, \quad (i,j=1,2,3),$$
$$ds^2 = g_{ij} dx^i dx^j, \quad (i,j=1,2,3),$$

are respectively the initial and additional metric parts and

$$2d\bar{s} \cdot d\bar{s} = 2g_{ij} \bar{d}x^i d\bar{x}^j$$ (3·3)

their interaction.

The metric tensor has three kinds of components, namely the barred $g_{ij}$, the circumflexed $\bar{g}_{ij}$ and the mixed $g_{ij}$. They can depend on the parameters

\[\hat{x}^i = \frac{dx^i}{dt}, \quad \hat{x}^i = \frac{d\bar{x}^i}{dt}\] (3·4)

or the ratios

$$d\bar{x}^1 : d\bar{x}^2 : d\bar{x}^3 : dx^1 : dx^2 : dx^3,$$ (3·4·1)

so that the space is a Finsler space referred to the six-dimensional elements of support

$$\left(x^i, \bar{x}^i; \hat{x}^i, \hat{x}^i\right).$$

Since no primary preference is assumed between the barred and circumflexed degrees of freedom, by a coordinate transformation, (3·1) can be reduced to

$$ds^2 = d\bar{s}^2 + d\bar{s}^2,$$ (3·1·1)

where the mixed term is absent, i.e.,

$$d\bar{s} \cdot d\bar{s} = 0, \quad g_{ij} = 0.$$ (3·5)

Whether (3·5) is assumed or not the quantities with the barred indices and those with the circumflexed indices can represent different kinds of physical objects with or without an interaction between them.

The covariant differentiation of a vector $X^i$ has the barred components

$$DX^i = dx^i + C^i_{j,k} X^j dx^k + \Gamma^i_{j,k} X^j dx^k + C^i_{j} X^j dx^k + \Gamma^i_{j,k} X^j dx^k + C^i_{j} X^j dx^k + \Gamma^i_{j,k} X^j dx^k + C^i_{j} X^j dx^k + \Gamma^i_{j,k} X^j dx^k.$$

Similarly for the circumflexed components. It should be noted that $DX^k$ need not vanish even if $X^k = 0$.

3·2. Partial oscillation. If we assume that the ratios

$$d\bar{x}^1 : d\bar{x}^2 : d\bar{x}^3$$ (3·5)

or the $\hat{x}^i$’s are fixed at each point as definite functions of the barred coordinates $x^i$’s,

$$\hat{x}^i = \xi^i (x^i),$$ (3·6)

then the same osculation processes as in §2 can be applied, hardly with any modification, to the barred degrees of freedom. It follows that

$$DX^i = dx^i + A^i_{j,k} X^j dx^k + C^i_{j,k} X^j d\bar{x}^k + \Gamma^i_{j,k} X^j d\bar{x}^k + C^i_{j} X^j d\bar{x}^k + \Gamma^i_{j,k} X^j d\bar{x}^k + A^i_{j} X^j d\bar{x}^k + C^i_{j} X^j d\bar{x}^k + \Gamma^i_{j,k} X^j d\bar{x}^k,$$ (3·7)
where
\[ A_{j}^{i} = C_{j}^{i} D_{l} \xi^{i} + \Gamma^{i}_{jl}, \]
(3.8)
\[ A_{j}^{i} = C_{j}^{i} D_{l} \xi^{i} + \Gamma^{i}_{jl}, \]
(3.6) being substituted in \( C_{j}^{i}, C_{j}^{i}, \Gamma^{i}_{jl}, \Gamma^{i}_{jl} \).

Obviously, these parameters \( A_{j}^{i}, A_{j}^{i}, C_{j}^{i}, C_{j}^{i} \)
depend on the unfixed degrees of freedom of the element of support, i.e., \( \xi^{i} \)'s or the ratios
\[ dx^{i} : dx^{i} : dx^{i}, \]
(3.9)
as well as on \( x^{i} \)'s and \( x^{i} \)'s.

We may restrict the problem by assuming
\[ X^{i} = 0 \]
and
\[ dx^{i} = 0, \]
(3.10) while the ratios (3.9) are preserved. Then (3.7) is reduced to
\[ DX^{i} = dX^{i} + A_{j}^{i} X^{j} dx^{i} + C_{j}^{i} X^{j} d\xi^{i}, \]
(3.7-1)
which has almost the same structure as the covariant differential (1-13) except that \( \xi^{i} \) and \( A_{j}^{i} \) are substituted for \( \xi^{h} \) and \( \Gamma^{i}_{jl} \).

If
\[ C_{j}^{i} \xi = 0, \text{ i.e. } \partial x^{i} g_{j} = 0, \]
(3.11) then
\[ DX^{i} = dX^{i} + A_{j}^{i} X^{j} dx^{i} + S_{j}^{i} X^{j} dx^{i}, \]
where
\[ A_{j}^{i} = A_{j}^{i}, \quad S_{j}^{i} = A_{j}^{i}. \]
(3.12)
Therefore, based on §2-3 we have

Lemma 3. The barred element of support can be made to be responsible for the distribution of dislocations.

Similarly, based on §2-4, we can also consider

Lemma 4. The barred element of support can be made, if it is preferred, to be responsible for the distribution of spins.

However, we can also consider another partial osculation in which \( l^{i} \)'s in place of \( l^{i} \)'s are given as fixed functions of \( x^{i} \)'s and proceed in a dual manner. We then obtain

Lemma 4'. The circumflexed elements of support can be made to be responsible for the distribution of spins.

3-3. Double osculation. Both the barred and circumflexed degrees of freedom can undergo the processes of osculation where both \( l^{i} \) and \( l^{i} \) are given as fixed functions of \( x^{i} \)'s.

We shall fix the ratios (3.9) at each point of the barred space by putting
\[ \dot{\xi}^{i} = \dot{\xi}^{i} (x^{i}) \]
(3.6-1)
where \( \dot{\xi}^{i} \) are definite functions of \( x^{i} \). Then both \( A_{j}^{i} \) and
\[ \dot{A}_{j}^{i} = C_{j}^{i} D_{l} \xi^{i} \]
become fixed at point \( x^{i} \). The covariant differential is now reduced to:
\[ DX^{i} = dX^{i} + A_{j}^{i} X^{j} dx^{i} + Z_{j}^{i} X^{j} dx^{i}, \]
(3.14)
where
\[ A_{j}^{i} = [A_{j}^{i}]_{x^{i} \rightarrow x^{i} (x^{i})}, \]
(3.15)
or
\[ DX^{i} = dX^{i} + A_{j}^{i} X^{j} dx^{i} + (\dot{S}_{j}^{i} + \dot{A}_{j}^{i}) X^{j} dx^{i} \]
(3.13-1)
where
\[ \dot{A}_{j}^{i} = [A_{j}^{i}]_{x^{i} \rightarrow x^{i} (x^{i})}, \quad x^{i} = x^{i} (x^{i}), \]
\[ \dot{S}_{j}^{i} = \dot{S}_{j}^{i}, \]
(3.16)
(3.17)
Thus finally, the six-dimensional Finsler space with the restrictions (3.10) and (3.1-6) has been reduced to a space of three-dimensional linear metric connexion with the parallelism defined by
\[ A_{j}^{i} = A_{j}^{i} + \dot{Z}_{j}^{i} \]
and the torsion tensor
\[ S_{j}^{i} = S_{j}^{i} + \dot{S}_{j}^{i}. \]

Owing to Lemmas 3 and 4', the latter consists of a dislocational torsion \( S_{j}^{i} \) and the magnetostrictive torsion \( \dot{S}_{j}^{i} \) (cf. reference [71]).

4. Curvature Features

The anomalous aspects of Finsler's space has more than one facet, one of which has been represented by the torsion tensor \( C_{j}^{i} \), \( A_{j}^{i} \) or \( S_{j}^{i} \) and another is summarized into the concept of intrinsic curvature. We shall review it from the point of view of the osculation.

4-1. Definition. The intrinsic curvature of a
space is defined by a failure along a small circuit in the manifold of elements of support. For any vector \( X^i \) the failure quantity for the parallel displacement is given by

\[
\Delta X^i = \Omega^i_{\ell} X^\ell, \quad (4.1)
\]

where

\[
\Omega^i_{\ell} = S^i_{\ell\lambda} D D^\lambda + P^i_{\ell\lambda} [dx^\lambda D^\ell ] + R^i_{\ell\lambda} [dx^\lambda dx^\ell], \quad (4.2)
\]

the coefficients \( R^i_{\ell\lambda}, P^i_{\ell\lambda}, S^i_{\ell\lambda} \) on the right-hand side being tensors defined as follows.

We have

\[
R^i_{\ell\lambda} = \partial_\lambda \Gamma^i_{\lambda\beta} - \partial_\beta \Gamma^i_{\lambda\lambda} + \Gamma^i_{\lambda\gamma} \Gamma^\gamma_{\lambda\lambda} - \Gamma^i_{\lambda\gamma} \Gamma^\gamma_{\lambda\beta}
\]

\[
\Gamma^i_{\lambda\beta} = \partial_\lambda \Gamma^i_{\beta\gamma} + \partial_\beta \Gamma^i_{\lambda\gamma} - \partial_\gamma \Gamma^i_{\lambda\beta}
\]

\[
- \partial_\lambda \Gamma^i_{\beta\gamma} \partial_\beta \partial_\gamma G^\lambda + \partial_\beta \Gamma^i_{\lambda\gamma} \partial_\gamma \partial_\lambda G^\beta
\]

\[
+ C^\gamma_{\lambda\beta} \partial_\beta \partial_\gamma G^\lambda
\]

\[
+ G^\gamma_{\lambda\beta} \partial_\beta \partial_\gamma G^\lambda \quad (4.3)
\]

where \( G^\lambda \) has been defined by (1.21) in §1.3 and

\[
G^\lambda_{\beta\gamma} = \partial_\beta \partial_\gamma G^\lambda (x, \dot{x}).
\]

We may not need to write down the expressions of \( P^i_{\ell\lambda} \) and \( S^i_{\ell\lambda} \). One can also define a tensor \( K^i_{\ell\lambda} \) related to \( r^i_{\ell\lambda} \) by

\[
K^i_{\ell\lambda} = K^i_{\ell\lambda} + C^\gamma_{\lambda\beta} K^\gamma_{\beta\gamma} H^\lambda. \quad (4.4)
\]

On performing an osculation, (4.2) is reduced to

\[
\Omega^i_{\ell} = L^i_{\ell\lambda} [dx^\lambda dx^\ell], \quad (4.5)
\]

where we have

\[
L^i_{\ell\lambda} = \partial_\lambda \Pi^i_{\ell\beta} - \partial_\beta \Pi^i_{\ell\lambda} + \Pi^i_{\ell\gamma} \Pi^\gamma_{\lambda\beta} - \Pi^i_{\ell\gamma} \Pi^\gamma_{\lambda\beta} \quad (4.6)
\]

with the parameters \( \Pi^i_{\ell\beta} \) defined in §2.2. The tensor \( L^i_{\ell\lambda} \) agrees with the so-called Riemann-Christoffel curvature tensor of the space of linear affine connexion.

The following relations should be noted

\[
R^i_{\ell\lambda} = - R^i_{\lambda\ell}, \quad R^i_{\ell\lambda} = - R^i_{\lambda\ell}, \quad (4.7)
\]

\[
K^i_{\ell\lambda} = K^i_{\lambda\ell}, \quad K^i_{\ell\lambda} + K^i_{\lambda\ell} + K^i_{\ell\lambda} = 0, \quad (4.8)
\]

\[
P^i_{\ell\lambda} = - P^i_{\lambda\ell}, \quad S^i_{\ell\lambda} = - S^i_{\lambda\ell}, \quad (4.9)
\]

4.2. Teleparallelism. If we consider an exceptional configuration in which \( \Delta X^i \) vanishes whatever the \( X^i \), then

\[
\Omega^i_{\ell} = 0 \quad (4.10)
\]

so that, from (4.2), we have

\[
R^i_{\ell\lambda} = - P^i_{\ell\lambda} D^{\lambda} = - S^i_{\ell\lambda} D^{\lambda} \quad (4.10.1)
\]

where \( l^m \) are given functions of \( x^k \).

Equation (4.10.1) can be regarded as partial differential equations for \( l^m \). If they are solved to give the distribution of \( l^m \) as functions of \( x^i \), a particular kind of osculation is obtained where the Riemann-Christoffel curvature tensor identically vanishes. This is what we call a teleparallelism.\(^{10}\)

The teleparallelism requires also the torsion tensor defined by (2.9) to be of a particular class. The procedures of assigning such a specific distribution of the three-dimensional torsion tensor as makes the resulting non-Riemannian space a space of teleparallelism has been called a perfect tearing.\(^2\) In our Finslerian approach, the perfect tearing consists of the processes of assuming such a specific distribution of elements of support as makes the osculating space a space of teleparallelism. We shall call the corresponding osculation the teleparallelism osculation.

4.3. Incompatibility. If the parameter \( t \) is treated as the time variable, then \( \dot{x}^k = d x^k / d t \) is the velocity. Then the average of the structural quantities over all possible directions of \( \dot{x}^k \) signifies a statistically statical condition.

If \( g_{\ell j} (x, \dot{x}) \) is assumed to be an even function of \( \dot{x}^i \), i.e., if

\[
g_{\ell j} (x, \dot{x}) = g_{\ell j} (x, - \dot{x}), \quad (4.11)
\]

then \( C_{\ell j k} \) is an odd function. From this follows

\[
\lim_{\dot{x}^k \to 0} C_{\ell j k} = 0, \quad (4.11.1)
\]

which is practically equivalent to the effect that the average of \( C_{\ell j k} \) for all the directions \( \dot{x}^i \) vanishes.

Since the ratios such as (1.2) are alone meaningful, the limit \( \dot{x}^i \to 0 \) does not imply that the Finslerian structure vanishes. Even if it is preserved its effects in the average can be restricted, so that the assumption (4.11) or (4.11.1) is valid. Hence, the limit symbol \( \lim \) hereafter means the averaging over all \( \dot{x}^k \), without implying \( \dot{x}^k = 0 \).

---

1) See [11] and [12].
2) See reference [13].
In particular we obtain, for the average values,

$$\lim_{\dot{x}_h \to 0} \Gamma^{ij}_{kjh} = \lim_{\dot{x}_h \to 0} \gamma^i_{kj} = \left[ \frac{i}{k} \right],$$

so that

$$\lim_{\dot{x}_h \to 0} R^{\dot{ijkl}}_{kjh} = \lim_{\dot{x}_h \to 0} K^{\dot{ijkl}}_{kjh} = \partial_k \left[ \frac{j}{j} \right] - \partial_h \left[ \frac{j}{ik} \right] + \left[ \frac{j}{ik} \right] \left[ \frac{m}{mk} \right] - \left[ \frac{j}{mk} \right] \left[ \frac{m}{ik} \right]. \quad (4.12)$$

This limiting case agrees with the Riemann-Christoffel curvature tensor of a Riemannian space with the Levi-Civita parallelism defined by the Christoffel three-index symbols $\left[ \frac{i}{k} \right]$ as the parameters of connexion. Let it be denoted by

$$\bar{K}^{\dot{ijkl}}_{kjh} = \lim_{\dot{x}_h \to 0} K^{\dot{ijkl}}_{kjh}. \quad (4.12)$$

The tensor $L^{\dot{i}}_{kjh}$ is reduced for this limit to

$$\bar{L}^{\dot{i}}_{kjh} = - \bar{K}^{\dot{i}}_{kjh} + \lim_{\dot{x}_h \to 0} \left( P^{\dot{i}}_{lkm} D_l \xi^m + S^{\dot{i}m}_{l} D_l \xi^m D_h \xi^n \right).$$

In the case of teleparallelism osculation, we have

$$\bar{L}^{\dot{i}}_{kjh} = 0,$$

where

$$\bar{K}^{\dot{i}}_{kjh} = - M^{\dot{i}}_{kjh} \quad (4.13)$$

In three-dimensional problems, the curvature tensor $R^{\dot{i}}_{jkh}$ has $3 \times 3$, and only $3 \times 3$, linearly independent components. Hence nothing is lost, even if we consider the contracted tensor

$$R^{\dot{ijkl}}_{jkh} = \bar{g}^{\dot{i}k} R^{\dot{i}h}_{jkh},$$

in place of $R^{\dot{i}}_{jkh}$. Therefore the teleparallelism relation $(4.10.1)$ can be

$$R^{\dot{ijkl}}_{jkh} = - \left( P^{\dot{i}km}_{\dot{h}} D_l \xi^m + S^{\dot{i}m}_{l} D_l \xi^m D_h \xi^n \right) \quad (4.10.2)$$

or

$$\bar{K}^{\dot{ijkl}}_{jkh} = - \frac{1}{2} \bar{g}^{\dot{ij}} \bar{g}^{\dot{jk}} = - T_{jkh} (j, k = 1, 2, 3) \quad (4.14)$$

where

$$\bar{K}^{\dot{ijkl}}_{jkh} = - M^{\dot{i}k}_{jkh},$$

\[\bar{K}^{\dot{ijkl}}_{jkh} = \bar{g}^{\dot{i}k} R^{\dot{i}h}_{jkh}, \]

\[T_{jkh} = M^{\dot{i}k}_{jkh} - \frac{1}{2} M^{\dot{i}h}_{jkh} \bar{g}^{\dot{ij}}. \]

For the limit $\dot{x}_h \to 0$,

$$e_{ij} = \frac{1}{2} (\bar{e}_{ij} - g_{ij})$$

is regarded as the strain in the ordinary sense, and the tensor

$$\bar{I}_{jkh} = \bar{K}_{jkh} - \frac{1}{2} \bar{g}_{jkh}$$

is often called the incompatibility tensor. Nothing is lost in it from the three-dimensional Riemann-Christoffel curvature tensor except in the special case in which the Riemannian space turns out to be a specific kind called the Einsteinian space, which is a space satisfying

$$\bar{K}_{jkh} = \frac{1}{2} \bar{g}_{jkh}.$$

The incompatibility tensor is constructed from the Riemannian metric tensor $\bar{g}_{jkh}(\delta^i = 0)$. But it needs to agree in statical (i.e. three-dimensional) problems with the quantity $T_{jkh}$ which has been constructed as the summary of the dislocational structure produced by a perfect tearing. In other words,

**Lemma 5.** By perfect tearing the Riemannian incompatibility of a material body in statical equilibrium is transformed into the torsion tensor field.

### 4.4. An approach to the Einsteinian formalism.

The formulae obtained in the foregoing can be extended to four dimensions by adopting

$$x^0 = t$$

as the fourth variable and assuming

$$ds^2 = \bar{g}_{ij} dx^i dx^j + \bar{g}_{00} dt dx^0 \quad (4.15)$$

as the fundamental metric form, where the statical limiting condition is assumed for the space part so that $(4.12)$ holds and $\bar{g}_{ij} (i, j = 1, 2, 3)$ do not depend on $x^0$ and $\delta^i (i = 1, 2, 3)$. Moreover, if $\bar{g}_{00}$ does not depend on the space coordinates $x^i$ and $\bar{g}_{ij}$ does not depend on $x^0$ then the purely space part of the curvature feature is not distinguished in structure from the three-
dimensional $\hat{K}_{\epsilon k l}$ and $\hat{K}_{\epsilon k}$. Hence (4.14) is preserved. Because it consists of components of a tensor equation it can be generalized to four dimensions

$$\hat{K}_{\epsilon\kappa}-\frac{1}{2}\hat{g}_{\epsilon\kappa} = -T_{\epsilon\kappa}(k, \lambda=0, 1, 2, 3) \quad (4.15)$$

in agreement with the general relativistic field equation where $T_{\epsilon\kappa}$ plays the rôle of the material-energy tensor.

Therefore, the general relativistic material energy tensor has been constructed by the perfect tearing of the Finslerian structure at a specific condition where the metric tensor assumes the form

$$\hat{g}_{\epsilon\kappa} = \begin{pmatrix} \hat{g}_{\epsilon\kappa} & 0 \\ 0 & \hat{g}_{\kappa\kappa} \end{pmatrix}.$$ 

Adopting the same structure as the limit $\varepsilon \to 0$ need not imply that all $\hat{\phi}^{k}$ vanish. It indicates the average effects similar to those of §4-3. The derivation of the Einsteinian field equation in this manner depends on the assumption of a statistically static space. However, non-static states are also derived by four-dimensional coordinate transformations.

At any rate, we have

**Lemma 5'**. The material energy tensor is a summary of more microscopic physical properties than are represented by the Riemannian metric at the limit (4.12), for the space part structure in regard to a certain coordinate system. The contents of the microscopic physics can be dislocations, spin distributions, etc.

Since the field equation has originally been derived by a variation from

$$\delta \int (\sqrt{g} + T\sqrt{g}) dX = 0$$

where

$$T = T_{\epsilon\kappa}g^{\kappa\kappa},$$

$$g = \det (g_{\epsilon\kappa}),$$

and $dX$ the four-dimensional space-time element, one can penetrate the dislocational and/or spin structure of the material-energy by varying the corresponding microscopic degrees of freedom lumped into $T_{\epsilon\kappa}$. Then, also the resistance or stresses corresponding to the spin and/or dislocational disturbances will automatically be introduced. There have been some investigations of more or less phenomenological nature in the sense that they assume rather than construct the corresponding geometrical concepts (e.g. [7], [10]). The present constructive approach will afford a natural introduction of these resisting quantities or generalized stresses.

**REFERENCES**


