Dual Dislocations and Non-Riemannian Stress Space

By

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双対転位と非リーマン応力函数空間

甘利俊一 景川耕字

＜概要＞

物質の塑性状態は非リーマン塑性論によって明らかにされる。そこでは歪変形空間の計量構造があるによって与えられ、増率が転位によって与えられる。この空間のEinsteinテンソルが歪の不適合度を与える。

これらは変形学的概念にもとづく量であるが、これらに対応する力学的概念をもとにして物質空間を構成することを考える。歪に対応する量は応力である。転位に対応する力学的な変、すなわち転位の増加に抵抗する抗力を表す物理量を新たに導入して、これを双対転位と呼ぶ。さらに不適合度の増加に対する抗力を表す量を調べると、これがBeltramiの応力函数テンソルに外ならないことが証明できる。

応力函数に基づいて計量を、双対転位に基づいて増率を定義すると、歪変形空間に双対な応力変形空間が得られる。この空間のEinsteinテンソルは応力になっている。この空間はSchaef erや皆川の応力函数空間を完全対応させて非リーマン空間に拡張したものになっている。この空間の構造を調べることによって、応力函数、双対転位および応力の間の関係が得られ、転位に関する方程式が導かれる。
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1. Introduction

The physical state of material including plastic defects has been clarified by non-Riemannian plasticity theory, by summarizing the defects in the structures of a material manifold (strain space). In the manifold, the metric is defined by the strain of the material, and the torsion tensor by the dislocation distribution. The incompatibility of the strain is then represented by the curvature or Einstein tensor due to the Levi-Civita parallelism of the space. The physical quantities appearing in the above are concerned about deformations of the material and the strain space is thus constructed upon them.

The physical state of material can also be specified by the dynamical or static quantities. These two methods of description are dual in the sense that the stored energy of the material is expressed by the product of a deformation quantity and its corresponding static quantity. The static quantity corresponding to the strain is the stress, as is well known, and it represents the physical actions resisting the growth of the strain. As the continuation of [2], the static quantity corresponding to the dislocation was introduced in [3]. It represents the physical actions resisting the growth of dislocations. We shall call it dual dislocation. Furthermore, we can prove that Beltrami's stress-function tensor is exactly the static quantity corresponding to the incompatibility, representing the physical actions resisting the growth of the incompatibility. Therefore, dually to the strain space, we can construct the stress space whose fundamental structures are defined by these static quantities.

The strain space is reduced to a Riemannian space by neglecting the torsion structure (dislocation). The Riemannian stress space dual to

1) See, e.g., [1]
the Riemannian strain space has been constructed by H. Schaefer [4]. The metric of the space is defined by Beltrami's stress-function tensor and the stress is represented by the curvature or Einstein tensor of the Levi-Civita parallelism. The space was investigated in detail by S. Minagawa [5], [6], and Prof. K. Kondo pointed out the importance of the difference of the orders existing between the strain and stress spaces, as well as their duality [7].

As the strain space can be considered from more microscopic viewpoint as a non-Riemannian space having the metric (strain), torsion (dislocation) and curvature (incompatibility) structures, so can be considered the stress space as a non-Riemannian space whose metric, torsion and curvature are determined by the stress-function, dual dislocation and stress. We shall investigate the structures of the non-Riemannian stress space. It will be proved that the stress space is a distant parallelism space, if the corresponding strain space is assumed to be a distant parallelism space by perfect tearing of the defects. The relation between the stress function and the dual dislocation will be obtained by this approach and their physical meanings will be clarified. The equilibrium equation governing dislocation distributions will also be derived in terms of the dual dislocations.

Recently, the torsion tensor was introduced into the stress space by R. Stojanovitch [8], E. Kroner [9] and S. Minagawa. There, the torsion represents the moment stress. Our approach is based on the duality that the corresponding quantities of both spaces are paired to give a scalar energy, and is different from theirs. However, some relations will be found between these spaces. It remains to be searched further as well as the problem to generalize our space to more general non-distant parallelism one.
2. Structures of Strain Space

A short summary will be given on the theory of distant parallelism strain spaces assuming small disturbances for simplicity's sake. For more detailed discussions on the subjects, refer to, for example, [10].

A small material piece represented by the vector line element $dx^i$ in the material manifold has been deformed because of the plastic defects existing in it. If we tear the element from the constraints of the surroundings, the deformation or strain of the small material piece can be removed by deforming the piece to the natural or perfect state without any internal stresses. Hence it can be mapped on a line element $(dx)^a$ of the natural state (in the case of the distant parallelism theory, we can adopt a perfect lattice as the natural state). Here $x^i (i = 1, 2, 3)$ are the orthogonal Cartesian coordinates denoting the position of a material point and $(dx)^a (a = 1, 2, 3)$ are the components of the line element measured by the orthogonal frame $(a)$ taken in the natural state. The $(dx)^a$ are linearly related to $dx^i$ by

$$(dx)^a = B^a_i dx^i, \quad dx^i = B^i_a (dx)^a, \quad (2.1)$$

where $B^a_i$ and $B^i_a$ are functions of $x^j$ and are mutually the inverse transformations.

Assuming that the disturbances are small, $B^a_i$ splits into two terms

$$B^a_i = \delta^a_i + \beta^a_i \quad (2.2)$$

where $\delta^a_i$ is the Kronecker delta and $\beta^a_i$ represents the small deformation occurring in the process of naturalization.

1) Einstein's summation convention is assumed throughout this paper.
The structures of the strain space are defined with reference to the naturalized state of the material elements as follows. The length $ds$ of a material element $dx^i$ is defined by the Euclidean length of the element $(dx)^a$ in the naturalized state, so that the strain space is equipped with a Riemannian metric $\varepsilon_{ji}$

$$ds^2 = \varepsilon_{ji} dx^i dx^j. \quad (2.3)$$

By simple calculations from the definition, the metric proves to be

$$\varepsilon_{ji} = \delta_{ji} - 2 \beta_{(ji)} \quad (2.4)$$

where

$$\beta_{ji} = \beta^a_i \delta_{a j}$$

and $( )$ means the symmetric part, i.e.,

$$\beta_{(ji)} = \frac{1}{2}(\beta_{ji} + \beta_{ij})$$

The ordinary parallelism in the natural state is also mapped to the strain space, giving the Euclidean connexion $\nabla^a_k$ to it. Neglecting the higher order terms, the parameters of the connexion are

$$\nabla^a_k = - \partial_k \beta^a_j, \quad (2.5)^1$$

where

$$\partial_k = \frac{\partial}{\partial x^k}$$

1) The difference between the covariant and contravariant characters of indices are disregarded, because it has been assumed that

$$\varepsilon_{ji} = \delta_{ji} + \text{small quantities.}$$

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The torsion tensor $S_{kj}^i$ of the space is, therefore, written as

$$S_{kj}^i = -\mathcal{A}(k^i j^i), \quad (2.6)$$

where $\mathcal{A}$ means the alternation, i.e.,

$$\mathcal{A}(k^i j^i) = \frac{1}{2} (\mathcal{A} k^i j^i - \mathcal{A} j^i k^i).$$

It is easy to verify that the space is distant parallelism, i.e., the Riemann-Christoffel curvature tensor $R_{kj}^i$ identically vanishes,

$$R_{kj}^i = 0. \quad (2.7)$$

However, the curvature tensor $K_{kj}^i$ due to the Levi-Civita parallelism of the metric does not vanish, and we have

$$K_{kj}^i = 2 \mathcal{A}(k^i j^i) j^i. \quad (2.8)$$

By virtue of (2.7), the identity

$$K_{kj}^i = 2 \mathcal{A}(k^i j^i) + 2 \mathcal{A}(j^i k^i) \quad (2.9)$$

holds.

Since the physical quantities are summarized in the geometrical structures, the former can be represented by the metric, torsion and curvature of the space as follows. The strain $e_{ji}$ is written in terms of the metric as

$$e_{ji} = \frac{1}{2} (\delta_{ji} - \delta_{ij}) = \beta_{(ji)}. \quad (2.10)$$

The dislocation (density) tensor $\delta_{ji}$, of which the first index refers to the direction of the dislocation lines and the latter to the direction of Burgers vectors, is merely a different expression of the torsion tensor.
\[ \lambda_{ji} = \frac{1}{2} \varepsilon_{jkl} S_{kli}, \quad (2.11) \]

where \( \varepsilon_{jkl} \) is Eddington's \( \varepsilon \) defined by

\[
\varepsilon_{jkl} = \begin{cases} 
1, & \text{when } (j, k, l) \text{ is an even permutation of } (1, 2, 3), \\
-1, & \text{when } (j, k, l) \text{ is an odd permutation of } (1, 2, 3), \\
0, & \text{otherwise.}
\end{cases}
\]

The incompatibility tensor \( J^{ij} \) of the strain is represented by \( K_{ijklj} \), as is well known in the theory of finite deformations,

\[
J^{ij} = \frac{1}{4g} \varepsilon^{iql} \varepsilon^{jmn} K_{klnan}
\]

\[
= -\varepsilon^{iql} \varepsilon^{jmn} \partial_n \partial_m \varepsilon_{qkl}.
\]

It is noteworthy that the \( J^{ij} \) is identical with the Einstein tensor when the dimension number is three. \( J^{ij} \) is a non-divergent tensor,

\[
\partial_e J^{ie} = 0, \quad (2.13)
\]

From (2.9), it follows that

\[
J^{ij} = \varepsilon^{ikl} \partial_l \lambda_{jil} + \varepsilon^{ikl} \partial_l \lambda_{jil} \quad (2.14)
\]

This connect the incompatibility with the dislocation distribution, since in the distant parallelism theory all the plastic defects are able to be torn in dislocations.

As a summary, the correspondences between geometrical structures and physical quantities in the strain space are given in Table 1.
Table 1  Strain Space

<table>
<thead>
<tr>
<th>Order of Differentiation</th>
<th>Geometrical Quantities</th>
<th>Physical Quantities</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>metric $\varepsilon_{ji}$</td>
<td>strain $\varepsilon_{ji}$</td>
</tr>
<tr>
<td>1</td>
<td>torsion $S_{kji}$</td>
<td>dislocation $\gamma_{ij}$</td>
</tr>
<tr>
<td>2</td>
<td>curvature $K_{lkji}$</td>
<td>incompatibility $J_{ij}$</td>
</tr>
</tbody>
</table>

$\rightarrow$ means relation by differentiation

3. Stored Energy and Static Quantities

Let us consider the energy $E$ stored in the material. It is assumed that the energy is given rise to only by the residual stresses due to the plastic defects and that no elastic stresses are applied from the outside. Since the space is distant parallelism, the dislocation and the incompatibility distributions are not independent (see (2.13)). Therefore, either of the dislocation and the incompatibility is sufficient to describe the plastic defects. Hence the energy can be expressed by the three ways, one in terms of the residual strains, another in terms of the dislocations and the third in terms of the incompatibilities.

Let $\varepsilon_{ji}$ and $J_{ij}$ be the residual strain and stress due to the plastic defects, respectively. Then the stored energy $E$ is
\[ E = \frac{1}{2} \int \epsilon_{ij} \sigma^{ij} \, dX, \]  

(3.1)

where

\[ dX = dx^1 dx^2 dx^3 \]

is a volume element. \( \sigma^{ij} \) represents the physical actions resisting the growth of \( e_{ij} \). Let us show it symbolically by

\[ \sigma^{ij} = \frac{\partial E}{\partial e_{ij}}. \]  

(3.2)

In terms of the incompatibility \( J^{ij} \), the \( E \) is also written as

\[ E = \frac{1}{2} \int J^{ij} \chi_{ij} \, dX. \]  

(3.3)

In this case \( \chi_{ij} \) means the physical actions resisting the growth of the incompatibility, and can be written symbolically as

\[ \chi_{ij} = \frac{\partial E}{\partial J^{ij}}. \]  

(3.4)

We can now show that the \( \chi_{ij} \) is identical with Beltrami's stress-function tensor, by which the stress is derived through differentiation,

\[ \sigma^{ij} = -\varepsilon^{ijk} \varepsilon_{mn} \partial_k \partial_m \chi_{ij}. \]  

(3.5)

The rôle and meaning of the stress-function tensor is clarified by (3.4) in the plasticity theory.\(^1\)

**Theorem 1.** The tensor \( \chi_{ij} \) representing the physical actions resisting the growth of the incompatibility \( J^{ij} \) is Beltrami's stress-function tensor by which the residual stress owing to the incompatibility is derived.

\(^1\) cf. \([11]\), \([12]\)
Proof. By virtue of (2.12),

$$E = \frac{1}{2} \int \mathbf{J}^i \mathbf{E}^i \, dX$$

is transformed into

$$E = \frac{1}{2} \int (-\epsilon^{ikl} \epsilon^{\alpha \beta} \partial_k \partial_\alpha \mathbf{E}^i_{\beta}) \mathbf{E}^i_{nl} \, dX$$

Integrating it twice by parts and assuming that the surface integrals vanish, it is further transformed into

$$E = \frac{1}{2} \int \epsilon^i_{\beta j} (-\epsilon^{ikl} \epsilon^{\alpha \beta} \partial_k \partial_\alpha \mathbf{E}^i_{\beta n} \mathbf{E}^i_{nl}) \, dX.$$ 

Comparing this with (3.1), we obtain

$$\xi^{ij} = -\epsilon^{ikl} \epsilon^{\alpha \beta} \partial_k \partial_\alpha \mathbf{E}^i_{\beta n} \mathbf{E}^i_{nl},$$

which proves the theorem.

Here it should be noted that \( \chi_{ji} \) can be determined only to within an additive term \( \mathcal{F} \) (where \( \mathcal{F} \) is an arbitrary vector field, because \( \mathbf{J}^i \) is non-divergent tensor,

$$\partial \mathbf{J}^i = 0. \quad (3.6)$$

Let us call the transformation of the stress-function tensor

$$\chi_{ji} \longrightarrow \chi_{ji}' + \mathcal{F}$$

the gauge transformation of the first kind. It can also be called the dual compatible transformation. Obviously the stress tensor does not change by the gauge transformation. It is well known that, by choosing a suitable gauge, Maxwell's or Morera's stress-function tensor can be derived from \( \chi_{ji} \). Moreover we can adopt such a gauge as satisfying

$$\partial_j \chi_{ji} = 0. \quad (3.7)$$

Schaefer's stress-function tensor is derived from this condition.
Let us denote by $\tilde{\alpha}^{ij}$ the physical actions resisting the growth of the dislocations. Since all the plastic defects are torn and split into dislocations, it holds that

$$E = \frac{1}{2} \int -\alpha_{ji} \tilde{\alpha}^{ij} \, dX.$$  \hspace{1cm} (3.8)

$\tilde{\alpha}^{ij}$ can also be denoted by

$$\tilde{\alpha}^{ij} = -\frac{2E}{\partial \alpha_{ji}}.$$  \hspace{1cm} (3.9)

We call the field $\tilde{\alpha}^{ij}$ the dual dislocation field, or $\tilde{\alpha}^{ij}$ the dual dislocation tensor.

**Theorem 2.** The dual dislocation is related to the stress-function tensor by

$$\tilde{\alpha}^{ij} = 2 \varepsilon^{ikl} \partial_k \chi_{lj}$$  \hspace{1cm} (3.10)

to within an additive term $\partial_j \gamma_i$.

**Proof.** By virtue of (2.13)

$$E = \frac{1}{2} \int \chi_{ji} \, dX$$

$$= \frac{1}{2} \int (\varepsilon^{ikl} \partial_k \chi_{ji} + \chi_{ik} \partial_k \chi_{lj}) \, dX$$

holds. Integrating by parts assuming the vanishing of the surface integrals and taking account that $\chi_{ji}$ is symmetric, we obtain

$$E = \frac{1}{2} \int -\alpha_{ji} (-2 \varepsilon^{ikl} \partial_k \chi_{lj}) \, dX.$$  

Comparing this with (3.8) and taking account of the non-divergent character of dislocations

$$\partial_j \alpha^{ji} = 0,$$  \hspace{1cm} (3.11)

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we can conclude that
\[ \bar{\alpha}^{ij} = 2 \varepsilon^{ikl} \partial_k \chi_{lj} \]
to within an additive term \( \partial_j \psi_i \), of which addition does not change the energy, for
\[ \int \partial_j \psi_i d\mathbf{X} = -\int (\partial_j \bar{\alpha}^{ji}) \psi_i d\mathbf{X} = 0. \]

The dual dislocation is determined except the gradient term \( \partial_j \psi_i \), where \( \psi_i \) is an arbitrary vector field. We call the transformation of the dual dislocation tensor
\[ \bar{\alpha}^{i\ell} \rightarrow \bar{\alpha}^{i\ell} + \partial_j \psi_i \]
the gauge transformation of the second kind. It is convenient to adopt the gauge by which
\[ \partial_j \bar{\alpha}^{i\ell} = 0 \]
(3.12)
holds. We call a couple of the gauge conditions (3.7) and (3.12) the Lorentz condition. Adopting the Lorentz condition, (3.11) exactly holds without any indefinite term. Moreover, the dual dislocation is non-divergent with respect to the first index, too,
\[ \partial_i \bar{\alpha}^{i\ell} = 0. \]
(3.13)
If another gauge is adopted, we must use
\[ \bar{\alpha}^{i\ell} = 2 \varepsilon^{ikl} \partial_k \chi_{lj} + \partial_j \psi_i \]
instead of (3.11), where \( \psi_i \) is the vector determined by the gauge conditions adopted there, and (3.13) does not necessarily hold.
4. Dual Dislocations

Let us investigate the relation between dislocations and dual dislocations. The problem obtaining the residual strain $\varepsilon_{ji}$ for the given dislocation field $\alpha_{ji}$ has been solved by many investigators.  

Let

$$
\varepsilon_{ji}(x) = a_{jk} \alpha_{kl} \delta_{k} \delta_{l}
$$

be the strain distribution caused by a unit dislocation at the origin whose direction is denoted by $\delta_{k}$ and whose Burgers vector is $\delta_{l}$. Hereafter the suffixes to $x$, $\xi$, etc. are suppressed for simplicity's sake. The $a_{jk}$ will easily be obtained from the classical elasticity theory. Using it, the residual strain for the given dislocation distribution $\alpha_{kl}(x)$ is obtained as

$$
\varepsilon_{ji}(x) = \int a_{ijkl}(x-\xi) \alpha_{kl}(\xi) \, d\xi,
$$

where

$$
d\xi = d\xi^{1} \, d\xi^{2} \, d\xi^{3}
$$

and the integration is taken over the whole material manifold. Since the stress and the strain are related to each other by

$$
\sigma_{ij} = E^{ijkl} \varepsilon_{kl},
$$

where $E_{ijkl}$ is Young's modulus tensor, the stored energy is directly written in terms of $\varepsilon_{ji}$ only by virtue of (4.2),

$$
\mathcal{E} = \frac{1}{2} \int E^{ijkl} \varepsilon_{ij} \varepsilon_{kl} \, d\xi \, d\eta \, d\zeta,
$$

(4.4)

1) See, e.g., Peach-Koehler's formula.
Defining the matter tensor $r^{\text{tsm}}(x)$, which depends on the kind of material only and does not depend on the physical state of it, by

$$r^{\text{tsm}}(x) = \int -E_{ijkl}a_{ikmn}(\xi) s_{jits}(\xi+\delta) d\Sigma,$$  (4.5)

(4.4) is simplified to

$$E = \frac{1}{2} \int -r^{\text{tsm}}(\xi-\eta) \alpha_{\kappa\lambda}(\xi) \alpha_{st}(\eta) d\Sigma d\Omega.$$  

As a consequence of this, it directly follows from the definition of $\Upsilon^{ij}_k(x)$ that

$$\Upsilon^{ij}_k(x) = \int r_{ijkl}(\xi-\eta) \alpha_{\kappa\lambda}(\xi) d\Sigma.$$  (4.6)

Hence it is shown that the dual dislocation is linearly related to the dislocation by the above integral equation. We can calculate the dual dislocation field from (4.6), when the dislocation distribution $\alpha_{\kappa\lambda}(x)$ is given.

By a similar reasoning, the method obtaining the stress-function tensor $\chi_{ji}$ from the incompatibility tensor $J_{ij}$ will be obtained as

$$\chi_{ji}(x) = \int \hat{s}_{ijlk}(\xi-\eta) J_{\ell\kappa}(\xi) d\Sigma.$$  (4.7)

It is noteworthy that the transformations

(4.6): $\alpha_{ji} \rightarrow \Upsilon^{ij}_k$

(4.7): $J_{ij} \rightarrow \chi_{ji}$

are transformations between non-divergent tensors (the Lorentz condition being assumed). The importance of such transformations has been pointed out in [13]. When the $r_{ijkl}(x)$ or $s_{ijkl}(x)$ vanishes sufficiently rapidly for large $x$ (mean-
ing that the effects of plastic defects are sufficiently localized), the integral relations (4.6) or (4.7) will be approximated by the simple averaged one
\[
\overline{\dot{\alpha} \dot{\gamma}(\vec{x})} = \overline{\dot{\gamma}^{jkl} \dot{\alpha}_{jk}^{\ell}(\vec{x})},
\]
or
\[
\dot{\alpha}_{j\dot{k}}(\vec{x}) = \overline{\dot{\gamma}_{j\dot{k}}^{\ell} \dot{J}^{\ell}(\vec{x})}.
\]

The force acting on a dislocation is given in terms of the dual dislocation tensor, since it represents the resistance against the growth of dislocations.

**Theorem 3.** The force acting on a unit dislocation \( \alpha_{ji} \) is
\[
F_{\dot{k}} = - \mathcal{E}_{(k \alpha_{i\dot{k}j})} \dot{\alpha}^{ji}. \tag{4.8}
\]

**Proof.** Since the dislocation field is non-divergent, a portion of a dislocation cannot move separately from the whole dislocation line. Therefore, it is equivalent to the appearance of the small dislocation loop \( C \) with Burgers vector \( b_{\dot{i}} \) for a portion of the dislocation \( \alpha_{ji} = d_{j\dot{i}} b_{\dot{i}} \) to move by \( dx^{\dot{k}} \) (see Fig. 1). The increment of the energy needed for this translocation is
\[
\Delta E = \oint_{C} \dot{\gamma}^{i} b_{\dot{i}} \dot{\alpha}^{\dot{i}j} dx^{\dot{j}}.
\]

Using Stokes' theorem, we obtain
\[
\Delta E = \int - \mathcal{E}_{(k \alpha_{i\dot{k}j})} b_{\dot{i}} d_{j\dot{k}}^{\dot{i}j} dx^{\dot{j}},
\]
where the integration is taken over the infinitesimally small area encircled by the loop $C$. Since the area can be represented by the bivector
\[
\delta [k, j] \delta x^k \delta x^j,
\]
we obtain
\[
\Delta E = - \partial_{(k} \epsilon_{i_1 i_2 i_3 i_4)}^\tau b^i \delta x^{(k} \delta x^{i_1} \delta x^{i_2} \delta x^{i_3} \delta x^{i_4)}
\]
Then the force is
\[
F_k = - \frac{\Delta E}{\delta x^k} = \partial_{(k} \epsilon_{i_1 i_2 i_3 i_4)}^\tau b^{i_1} \delta x^{i_2} \delta x^{i_3} \delta x^{i_4)}
\]
proving the theorem.

We have so far considered the effects of plastic defects only and have not taken the applied stresses into account. Here is investigated the effects of the applied stress $\tau^{i_1 i_2}$ to the dislocations.

When a dislocation $d_1 b_1$ moves in the direction $\delta x^k$, the atoms located above the slip surface
\[
\delta S_1 = \epsilon_{i_1 i_2 i_3 i_4} \delta x^i \delta x^j
\]
are displaced by $b_1$ relatively to those below the surface. Under the interaction with the applied stress $\tau^{i_1 i_2}$, these displacements require the energy
\[
\Delta E = \tau^{i_1 i_2} \delta S \delta x^k b_i
\]
\[
= \tau^{i_1 i_2} \epsilon_{i_1 i_2 i_3 i_4} \delta x^k b_i \delta x^j.
\]
Therefore, the force
\[ -\frac{\Delta E}{dx} = -\varepsilon_{kji} \tau^{\ell i} \]
acts on the dislocation \( \delta \ell \) according to the applied stress \( \tau^{\ell i} \). Hence the equilibrium condition of the dislocations is given by
\[ \partial_{(k \alpha | \ell | j)} = -\varepsilon_{kji} \tau^{\ell i} \]  
(4.9)

In the actual material, however, this equation may not necessarily be satisfied, because some frictional forces exist in it owing to the lattice structure.

As a summary, we have

**Theorem 4.** The equations governing the dislocation distribution are
\[
\begin{align*}
\partial_{j} \delta_{j} & = 0 \\
\partial_{(k \alpha | \ell | j)} = -\varepsilon_{kji} \tau^{\ell i},
\end{align*}
\]  
(4.10)

where \( \tau^{\ell i} \) is the applied stress.

5. Structures of Non-Riemannian Stress Space

The fundamental structures of the strain space, i.e., the metric, torsion, and curvature, are defined in reference to the strain, dislocation, and incompatibility. Dually to the above, we can construct the space whose metric, torsion and curvature represent the stress function, dual dislocation and stress, respectively. This is obviously a generalization of Schaefer-Minagawa's Riemannian stress space [4], [5].

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Let us define the metric tensor $\bar{g}_{ji}$ in a similar manner as (2.4) by

$$
\bar{g}_{ji} = \delta_{ji} - 2 \chi_{ji},
$$

(5.1)

where $\chi_{ji}$ is the stress-function tensor and is obviously symmetric. The torsion tensor $\bar{S}_{kji}$ can similarly be defined by the dual dislocation tensor (cf. (2.11)),

$$
\bar{S}_{kji} = \frac{1}{4} \varepsilon_{kjl} \bar{\chi}_{li}.
$$

(5.2)

Then the stress tensor $\sigma_{ij}$ is given by the curvature or Einstein tensor of the Levi-Civita parallelism with respect to the metric (5.1),

$$
\sigma_{ij} = \frac{1}{4} \varepsilon_{ikl} \varepsilon_{jmn} \bar{\kappa}_{k \ell mn} = - \varepsilon_{ikl} \varepsilon_{jmn} \partial_{k} \partial_{m} \chi_{\ell l}.
$$

(5.3)

This corresponds to (2.12) in the strain space. The stress tensor is symmetrical and automatically satisfies the non-divergence condition

$$
\partial_{i} \sigma_{ij} = 0,
$$

(5.4)

which physically means the equilibrium condition of forces.

The parameters of the Euclidean connexion are introduced in the space by

$$
\bar{\Gamma}^{c}_{k j} = \left\{ \bar{\Gamma}^{c}_{k j} \right\} + S^{c}_{k j} + 2 \bar{S}_{c} (k j),
$$

(5.5)

(see, e.g., p. 132 of [14]). Here $\left\{ \bar{\Gamma}^{c}_{k j} \right\}$ is the Christoffel three-index symbol of the second kind concerning the metric $\bar{g}_{ji}$.
\[
\{ \bar{e}^i \}_{j} = \frac{l}{2} \bar{e}^m (\partial_k \bar{\bar{g}}_{mj} + \partial_j \bar{\bar{g}}_{km} - \partial_m \bar{\bar{g}}_{kj}). 
\] (5.6)

By simple calculations assuming the Lorentz condition, we see that (5.5) reduces to

\[
\bar{\Gamma}^i_{kj} = -\partial_k \chi_{ji},
\] (5.7)

which corresponds to (2.5) in the strain space.

The strain space has been assumed to be distant parallelism. We shall see from the following theorem how this assumption restricts the stress space corresponding to it.

**Theorem 4.** The stress space corresponding to the distant-parallelism strain space is also distant parallelism, when the Lorentz condition is assumed.

This is proved by

\[
\bar{R}^{lk}_{jki} = 0,
\] (5.8)

which directly follows from (5.7). From the theorem, the stress space defined here proves to be obtained by the perfect tearing of the Riemannian stress space. The non-divergent character of the dual dislocation originates from the vanishing of the Riemann-Christoffel curvature tensor, as can easily be shown by Bianchi's identity

\[
\partial_\ell (\bar{\bar{S}}^{k}_{ji})^\ell = \bar{R}^{\ell}_{\ell kji}^\ell = 0.
\] (5.9)

Dually to (2.9), the curvature \( \bar{K}_{kji} \) due to the Levi-Civita parallelism is connected with the torsion \( \bar{S}^{k}_{ji} \) by

\[
\bar{K}_{kji} = 2 \bar{\partial}_{[\ell} \bar{S}^{\ell}_{jik]} + 2 \partial_{(\ell} \bar{S}^{\ell}_{jik)}^\ell,
\] (5.10)
and so is the stress with the dual dislocation by

\[ 4 \sigma^j = \epsilon^{jlm} \partial_x \alpha^{lm} + \epsilon^{jlm} \partial_x \alpha^{lm}, \]  

(5.11)
corresponding to (2.14).

In the material manifold, the natural state of the stress function can also be considered. Let a material vector element \( dx^i \) be mapped to \( (dx)^\alpha \) in the natural state of the stress-function metric by

\[ (dx)^\alpha = B_i^\alpha dx^i. \]  

(5.12)

Then the natural state is specified by the mapping tensor

\[ B^\alpha_i = \delta^\alpha_i + \chi^\alpha_i, \]  

\[ \chi^\alpha_i = \chi_{ij} \delta^\alpha_j. \]  

(5.13)

In the non-holonomic reference frame \( (\alpha) \) referred to the natural state, the metric reduced to

\[ g_{\alpha \beta} = \delta_{\beta \alpha}, \]  

(5.15)

and the torsion can be equated to the non-holonomic object

\[ T_{\beta \gamma} = \delta_{\gamma \beta}. \]  

(5.16)

Since the \( \chi_{ji} \) is a symmetric tensor, no rotations but pure deformations take place in the releasing processes of the material elements to the natural state, differing from the case of the strain naturalization.
The structures of the stress space is summarized in the following table.

**Table 2 Stress Space**

<table>
<thead>
<tr>
<th>Order of Differentiation</th>
<th>Geometrical Quantities</th>
<th>Physical Quantities</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>metric $g_{ji}$</td>
<td>stress function $\chi_{ji}$</td>
</tr>
<tr>
<td>-1</td>
<td>torsion $S_{kji}$</td>
<td>dual dislocation $\bar{\chi}_{ji}$</td>
</tr>
<tr>
<td>0</td>
<td>curvature $K_{ki}$</td>
<td>stress $\sigma_{ij}$</td>
</tr>
</tbody>
</table>

The interrelation of the strain and stress spaces are shown in Table 3.

**Table 3 Interrelation of the Strain and Stress Spaces**

<table>
<thead>
<tr>
<th>Order of Differentiation</th>
<th>Strain Space</th>
<th>Stress Space</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td></td>
<td>stress function $\chi_{ji}$</td>
</tr>
<tr>
<td>-1</td>
<td></td>
<td>dual dislocation $\bar{\chi}_{ji}$</td>
</tr>
<tr>
<td>0</td>
<td>$\varepsilon_{ij}$ strain</td>
<td>stress $\sigma_{ij}$</td>
</tr>
<tr>
<td>1</td>
<td>$\alpha_{ij}$ dislocation</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$\gamma_{ij}$ incompatibility</td>
<td></td>
</tr>
</tbody>
</table>

----- correspondence by space structure
      correspondence by energy

$\Rightarrow$ differentiation

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Both spaces are in two-fold duality, one being geometrical denoted by the dotted lines and the other being physical denoted by the full lines. By the former duality, the quantities of the differentiation orders \( k \) and \( k - 2 \) \( (k = 0, 1, 2) \) correspond to each other, while by the latter duality the quantities of the differentiation order \( \pm k \) are mutually coupled, as is the case with the duality in algebraic topology. It is interesting that the dislocation and dual dislocation correspond to each other by both dualities.

The equation governing the primal and dual dislocations are rewritten using the torsion tensors as

\[
\mathcal{D}_{[k} \overline{S}_{k]ij} : \mathbf{e} = 0, \quad (5.17)
\]

\[
4 \mathcal{D}_{[k} \overline{S}_{k]ij} = -\varepsilon_{k \bar{m}n} \varepsilon_{k \bar{j}m} \tau_{\bar{m}n}, \quad (5.18)
\]

The latter equation is rewritten using the Einstein tensor (stress tensor)

\[
\overline{\mathcal{D}}^{ij} = -\overline{\mathcal{T}}^{ij},
\]

because \( \mathcal{X}^{ij} \) is symmetric, which obviously has the same form as the Einstein gravitational equation of general relativity.

Introducing a new tensor \( \overline{S}'_{lmn} \) by

\[
\overline{S}'_{lmn} = \frac{1}{4} \varepsilon_{kj} \varepsilon_{lm} \overline{S}_{kji}, \quad (5.20)
\]

the equations are transformed into

\[
S_{kji} = \mathcal{D}_{k \beta j} \mathcal{R}^{\beta ij},
\]

\[
\mathcal{D}_{[k} S_{k]} \mathbf{e} = 0,
\]

\[
\mathcal{D}_{[k} \overline{S}'_{k]mn} = \tau_{km},
\]

\[
\overline{S}'_{klm}(\omega) = \chi_{ij m}(\omega - \xi) S_{jkm}(\xi) d\overline{\xi},
\]

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which obviously correspond to the Maxwell equations, if the dimension number is enlarged to four by introducing the time coordinate and taking into account of moving dislocations as has been studied in [2]. This is the extension of the electromagnetic analogy obtained in [15] to the non-isotropic case.

References


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