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A Geometrical Theory
of
Moving Dislocations and Anelasticity

by
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運動する軸位と非弾性の場の幾何学的理論

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＜概要＞

軸位を非リーマン幾何学の概念として把握する理論は近藤一夫教授によって一般的な形で発展され（たとえば [1], [2],[3] を見よ）ついで Bilby や Kröner によって遠隔平行性という束縛された形で提出された（[4],[5],[6] 等を見よ）。


軸位の運動は物質に塑性変形をひき起す。この運動と歪の不適合度の変化との関係を示し、また Eckart が現象論的に導入した「非弾性テンソル」[10]（自然計量の変化率を示す）を軸位の運動、軸位場の変化率より導き出し、その実体を明らかにする。
A Geometrical Theory

of

Moving Dislocations and Anelasticity

By

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Introduction

The imperfection of crystals called dislocation was first recognized by Prof. K. Kondo to have the character of the torsion tensor in non-Riemannian material space (see, e.g., [1]). The non-Riemannian theory including an elaborated theory of yielding was then developed remarkably by him ([2], [3]). After a while, B. A. Bilby and E. Kroner also independently analyzed continuously distributed dislocations using the concept of torsion within the framework of the assumption of distant parallelism (see, e.g., [4], [5] and also [6]).

In continuation of these investigations, the geometrization will be extended to cover not only fixed but also moving dislocation fields. We shall find that the motion of dislocations can be represented by the torsion tensor in a four-dimensional material manifold. By classifying the components of the torsion tensor, the types of the motion of dislocations as well as the types of dislocations themselves will be shown to be spontaneously classified. We shall derive the relation between dislocation distributions and their motions.

In D-XI [7], the motions of dislocations as well as dislocation pairs have already been analyzed and classified.\(^1\) The present treatment shows a close connexion with it and can be regarded as another approach to the same problem. But in this paper, the distant parallelism is assumed based of the mathematical formality of the virtual perfect tearing pointed out in D-X [8], while it is not assumed in D-XI. Hence some kinds of imperfections which would be represented by the Riemann-Christoffel curvature tensor, have been transformed into, and summarized in, the régime of torsion in this paper. The two standpoints will be combined into one, if a more general viewpoint such as the geometry of projective connexion, is introduced, as has been suggested by Kondo [9].

As dislocations move, the plastic state of material changes, giving rise to a certain relation between the change of the incompatibility of strains and the motion of dislocations. There is an anelasticity tensor which has been introduced phenomenologically by C. Eckart in

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1) Some of the references, e.g. [7], [8], etc., are in the press, to be published in the third volume of the RAAC Memoirs.
order to represent the rate of the change of the natural metric, i.e. the change of the plastic state [10]. We shall show that the same tensor can be derived from the rate of the change of the dislocation field, and the character of the anelasticity will be clarified therefrom.
1. Construction of Four-Dimensional Material Space

In order to obtain the geometrical terminology suitable for expressing the role of moving dislocations, a four-dimensional space will be constructed for a crystal including moving imperfections. The construction is done in a manner quite similar to that which has been adopted in the construction of the three-dimensional non-Riemannian material space\(^1\), so that the space handled in the following is non-Riemannian. In this paper, however, small disturbances are assumed and the distant parallelism approach is resorted to merely for simplicity's sake as said above. A more general non-Riemannian approach is also feasible and it will afford a natural extension of the present viewpoint.

1.1. Four-dimensional material manifold. Each point of a crystal including moving imperfections can be marked by the four-dimensional Cartesian coordinates of the position-time

\[ x^\nu, \quad \nu = 0,1,2,3 \]

which it occupies in real space-time, where \( x^0 = t \) denotes the time coordinate and \( x^\nu's ((\nu = 1,2,3)) \) denote the space coordinates. In other words, a point \( x^\nu ((\nu = 0,1,2,3)) \) means a location \( x^\nu ((\nu = 1,2,3)) \) in real space at time \( x^0 = t \). We can regard the imperfect crystal the points of which are marked by these coordinates as a four-dimensional manifold.

Since the crystal is imperfect, it is distorted, in the sense that some strains exist in the space or the crystallographic axes are distorted from those of a perfect crystal, and also that each point of the crystal is moving, i.e., it changes its attribute in regard to the time. We cannot relax all these distortions without tearing the material, for it is plastically deformed and including dislocations.\(^2\)

In order to the distorted state of the crystal, let us tear it into an aggregate of small material-times

\(^2\) See D-IX [11], D-XI [7].
\(^3\) Since we are handling a continuum theory, each torn piece is assumed to be infinitesimally small but to contain a sufficiently large number of atoms or crystal points.
We can regard \((dx)^\alpha\), the components of the vector \(dx^\alpha\), with reference to a new reference frame, which we call the natural frame. We can regard \((dx)^\alpha\) as the components of the vector \(dx^\alpha\) with reference to a new reference frame, which we call the natural frame.

and relax each piece by taking off all the distortion from it. Then each piece becomes a perfect crystal. We refer to this aggregate of small relaxed material-times as being in the "natural state", and we call the tearing and relaxing process "naturalization".

With the naturalization process, an infinitesimal vector line-element \(dx^\alpha\) in the distorted crystal is brought to \((dx)^\alpha\) \((a = 0,1,2,3)\) in the natural state, where \(a\) denotes a Cartesian coordinate system (see Fig. 1). We call the frame \(a\) the natural frame, which is generally non-holonomic in correspondence with the tearing process.

It can be assumed that \(dx^\alpha\) and \((dx)^\alpha\) are linearly related

\[
\begin{align*}
(dx^\alpha) &= B^\alpha_\beta (dx)^\beta, \\
(dx)^\alpha &= B_\alpha^\beta (dx^\beta),
\end{align*}
\]

(1.1)

where \(B_\alpha^\beta B^\beta_\gamma = \delta_\alpha^\gamma, B^\alpha_\beta B^\beta_\gamma = \delta_\alpha^\gamma\) holds. \(B^\alpha_\beta\)'s are functions of \(x^\alpha\).

If the torn pieces in the natural state are rearranged, then all the corresponding crystallographic axes will become parallel. That is a permissible process.\(^1\) By this rearrangement, we can determine \(B^\alpha_\beta\) uniquely, but it means to introduce distant parallelism assumption.\(^2\) A set of vectors \(\xi_\alpha\) \((a = 0,1,2,3)\) can be associated with each point of the distorted material-time as a basis of the natural frame \(a\). Obviously the \(x\)-th component of \(\xi_\alpha\) is \(B^\alpha_\beta\). The spatial set \(\xi_\alpha\) \((a = 1,2,3)\) of \(\xi_\alpha\)'s can be determined by referring to the crystallographic axes, whereas \(\xi_\alpha\) can be determined by referring to the velocity \(v\) or the trace of the motion of the material point. The trace of the motion of a material point may be called "world-line" in a little modified sense than in special relativity theory.

Here let us determine \(\xi_\alpha\) or the transformation tensor \(B^\alpha_\beta\), by which the distorted state of material is represented.

\(^1\) This process is clarified by Hondo as the perfect tearing and refurbishment \([11, 12]\).

\(^2\) See \([12]\).
Splitting (1.1) into the spatial and time terms, we have

\[
\begin{align*}
\frac{dx^\tau}{\tau} &= B^\tau_\tau (dx)^\tau + B^\tau_\tau d\tau, \\
\frac{dt}{\tau} &= B^\tau_\tau (dx)^\tau + B^\tau_\tau d\tau,
\end{align*}
\]

(1.2)

where \( \tau \) and \( \tilde{\tau} \) run over the spatial coordinates alone, i.e. \( \tau, \tilde{\tau} = 1,2,3 \), and we put \( dx^\tau = dt \) for the \( \tau \)-frame, \( dx^\tilde{\tau} = dt \) for the \( \tilde{\tau} \)-frame. We can see that a spatial vector \( (dx)^\tau = (0,0,0) \) in the natural state is \( \tau \)

also mapped to a spatial vector \( dx^\tau = (dx^\tau, 0) \) of the deformed state (see Fig.2). Hence putting \( dt = d\tau = 0 \) in (1.2), we have

\[
B^\tau_\tau = 0 \quad (\tau = 1,2,3)
\]

and we see that \( B^\tau_\tau \) represent the spatially deformed state at an instant \( t \). Notice the assumption (1.3) which indicate that the time is not essentially affected by the naturalization or tearing, hence the special relativistic formalism is not assumed. Next, we see that a time-vector \( (dx)^\tau = (0,0,0, dt) \) in the natural state is mapped on the trace of the crystal point, i.e. the world-line of the point in the deformed state. Hence it is mapped to \( dx^\tau = (v^\tau dt, v^\tau dt, v^\tau dt, dt) \), where \( v^\tau \) is the (spatial) velocity of the material point. Putting (1.2), we then have

\[
\begin{align*}
\frac{dx^\tau}{\tau} &= B^\tau_\tau d\tau, \\
\frac{dt}{\tau} &= B^\tau_\tau d\tau,
\end{align*}
\]

from which

\[
\begin{align*}
B^\tau_\tau &= v^\tau, \\
B^\tau_\tau &= 1,
\end{align*}
\]

\( \tau = 1,2,3 \)

holds. Putting \( B^\tau_\tau = \delta^\tau_\alpha + f^\tau_\alpha \), we have
where $\beta^\alpha_{\beta} = v^\alpha$, $\beta^\alpha_{\alpha} = 0$ and $\delta^\alpha_{\beta}$ is Kronecker's delta.

Since the distortion is assumed to be small, we can neglect the higher order terms of $\beta^\alpha_{\beta}$ and $v^{\alpha 1}$. Then the inverse transformation tensor $B^b_{\alpha}$ can be calculated as

$$B^b_{\alpha} = \frac{1}{b} \begin{pmatrix} \delta^b_{\alpha} & -\beta^b_{\alpha} & 0 \\ \delta^b_{\alpha} & \delta^b_{\alpha} & \beta^b_{\alpha} \\ 0 & 0 & 1 \end{pmatrix}, \quad (1.4 1)$$

where

$$\beta^b_{\alpha} = \delta^b_{\alpha} \delta^b_{\alpha} \beta^b_{\alpha},$$

$$v^b = \delta^b_{\alpha} v^\alpha.$$

### 1.2. Metric tensor and strain tensor

Let us introduce a metric into the material manifold. We first consider the case of a perfect crystal, i.e. a crystal in the natural state. Since the frame $(a)$ has been fixed to be orthogonal in space, the square of the real length $ds^2$ of a spatial vector $(dx)^a = (dx)^x, 0)$ is written as

$$ds^2 = \epsilon^a_{\alpha} (dx)^a (dx)^b = (dx)^x + (dx)^y + (dx)^z,$$

1) The velocity $v^b$ of a lattice point is considered to be small compared with the sound velocity. Here we adopt such a unit for time that the order of the sound velocity becomes 1.
which may be regarded, if need be, the spatial part of a metric to be
associated, for convenience's sake, with the four-dimensional
space-time with \((dx)^2 = dt^2\) as the four coordinates. In
other words, we can, for convenience sake, introduce the
metric tensor \(g_{\alpha\beta}\) having the following form

\[
g_{\alpha\beta} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad (1.5)
\]

where the components for \(a, b \neq \bar{\alpha}, \bar{\beta}\) are to be studied
further. Owing to the arbitrary character of the extension
of the metric, some assumptions may be introduced in
regard to the cross components of \(g_{\alpha\beta}\) relating to space
and time such as \(g_{\underline{\alpha} \underline{\xi}}\).

Assumption. As far as the metric characteristics
are concerned, it is impossible to distinguish
either between the positive and negative direction
of time keeping the space-part fixed, or between
the positive and negative direction of a space
vector keeping the time part fixed.

From this assumption, the lengths of vectors \((dx)^\bar{\alpha}, dt\)
and \((dx)^\alpha, - dt\) should equally be defined. Hence we
cannot but define \(g_{\alpha\bar{\alpha}} = g_{\bar{\alpha}\alpha} = 0\). The time component \(g_{\underline{\alpha} \underline{\alpha}}\)
is still indefinite. Putting \(g_{\underline{\alpha}\underline{\alpha}} = c\), the metric tensor
of the four-dimensional material space is defined by

\[
g = \begin{pmatrix}
1 & 0 & 0 & c \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & c
\end{pmatrix}, \quad (1.6)
\]
for a perfect crystal. If the indefinite constant $c$ is positive, the space is Euclidean, and if $c$ is negative, the space is Minkowskian. There is yet no physical reason to determine $c$. The constant $c$, therefore, remains indefinite throughout this paper.

The length $ds$ of a line element $dx^\kappa$ for the space of a deformed crystal is defined by the length of the corresponding element of a perfect crystal after naturalization. Let us call the metric the natural metric. Since $dx^\kappa$ corresponds to $(dx)^\kappa = B_\kappa^\lambda dx^\lambda$, we have

$$ds^2 = g_{\kappa\lambda} (dx)^\kappa (dx)^\lambda = g_{\kappa\lambda} B_\kappa^\rho B_\lambda^\rho dx^\rho dx^\lambda,$$

and the natural metric tensor can be written as

$$g_{\lambda\kappa} = g_{\kappa\lambda} B_\kappa^\rho B_\lambda^\rho$$

in the $\kappa$-frame.

Substituting (1.4.1) in (1.7), the metric tensor is rewritten as

$$g_{\lambda\kappa} = \delta_{\lambda\kappa} - 2 \beta_{\lambda\kappa}$$

where $\beta_{\lambda\kappa}$ is the symmetric part of $\beta_{\lambda\kappa}$, i.e. $\beta_{\lambda\kappa} = \frac{1}{2} (\beta_{\lambda\kappa} + \beta_{\kappa\lambda})$.

The local characteristics of the distorted state, i.e. the distorted state of the neighbourhood of a point of the crystal, is represented by the difference between the squared natural length $ds^2 = g_{\lambda\kappa} dx^\kappa dx^\lambda$ and the squared distorted length $\tilde{ds}^2 = \tilde{g}_{\lambda\kappa} dx^\kappa dx^\lambda$, where

$$\tilde{g}_{\lambda\kappa} = \delta_{\lambda\kappa} - 2 \beta_{\lambda\kappa}$$

and the matrix $\beta$ is symmetric.
Therefore, we define the strain tensor $\varepsilon_{\lambda\kappa}$ by

$$
\varepsilon_{\lambda\kappa} \overset{\text{def}}{=} \frac{1}{2} \left( \tilde{\varepsilon}_{\lambda\kappa} - g_{\lambda\kappa} \right),
$$

(1.9)

which shows the local distortion of the material. Using (1.8), we have

$$
\varepsilon_{\lambda\kappa} = \frac{1}{2} \left( \beta_{(\lambda\kappa)} - \frac{1}{2} u^\nu u^\nu \right),
$$

(1.10)

of which the spatial components $\varepsilon_{\lambda\kappa} = \beta_{(\lambda\kappa)}$ exactly coincide with the ordinary strain tensor defined in the threedimensional material manifold.

1.3. Metric affine connexion with distant parallelism. In order to investigate more macroscopic characteristics of the distortion than is shown by the metric or strain tensor, we introduce an affine connexion by which the relation between the distorted states of the neighbourhoods of around two neighbouring points will be clarified. This is closely connected with the non-holonomic character of the natural frame (a).

An affine connexion can be introduced by defining parallelism between two vectors $p^\lambda$ at $O(x^\lambda)$ and $p^\lambda + dp^\lambda$ at $O'(x^\lambda + dx^\lambda)$, by saying that they are parallel in the ordinary sense after being mapped on the perfect crystal in the natural state. This means that all the corresponding crystallographic axes are defined to be parallel in the spatial part and all the lines representing the traces of the motions of crystallographic points which can be brought to the direction parallel to the time direction by the mapping or the naturalization are parallel, i.e., all the world lines are parallel. If we construct a four-dimensional lattice from the world-picture of a three-
dimensional lattice by digitalizing the time, we may say that all the corresponding lattice axes are parallel, although such does not generally hold without assuming the distant parallelism.

Since the ordinary parallelism holds in the map corresponding to the perfect tearing, we put

\[ \Gamma^a_{cb} = 0 \]  \hspace{1cm} (1.11)\]

in the natural frame (a), defining the tearing. Since the relation

\[ \Gamma^a_{[cb]} = S^a_{cb} - \Omega^a_{cb} \]  \hspace{1cm} (1.11)\]

holds in any non-holonomic frame, where \( \Omega^a_{cb} \) is the non-holonomic object defined by

\[ \Omega^a_{cb} \overset{\text{def.}}{=} B^a_{[c} B^b_{b]} \partial^\gamma B^a_{\gamma} \]  \hspace{1cm} (1.11)\]

(1.11) means to define the torsion tensor \( S^a_{cb} \) by the non-holonomic object of the frame (a) by

\[ S^a_{cb} \overset{\text{def.}}{=} \Omega^a_{cb} \]  \hspace{1cm} (1.12)\]

From (1.11), we have

\[ \Gamma^a_{\mu\lambda} = B^a_{\mu} \partial^\gamma B^a_{\lambda} \]  \hspace{1cm} (1.13)\]

and

\[ S^a_{\mu\lambda\kappa} = B^a_{\mu} \partial^\gamma B^a_{\lambda\kappa} \]  \hspace{1cm} (1.14)\]

which is rewritten as

\[ S^a_{\mu\lambda\kappa} = - \partial^\gamma B^a_{\lambda\kappa} \]  \hspace{1cm} (1.15)\]

where \( \partial^\gamma = \frac{\partial}{\partial x^\gamma} \).

Obviously this space has distant parallelism, i.e., the Riemann–Christoffel curvature tensor \( R^a_{\mu\lambda\kappa} \) defined by

\[ R^a_{\mu\lambda\kappa} = \frac{1}{2} \left( \Gamma^a_{\mu\lambda \gamma} \Gamma^\gamma_{\kappa} - \Gamma^a_{\mu\kappa \gamma} \Gamma^\gamma_{\lambda} \right) \]

\[ \text{\footnotesize{1}} \] \( \epsilon \) and \( \bar{\epsilon} \) means "alternation", for instance, \( \Gamma^a_{[cb]} = \frac{1}{2} \left( \Gamma^a_{cb} - \Gamma^a_{bc} \right) \).
\[ R_{\nu \mu \lambda \kappa} = 2 \varepsilon_{\nu} \mu \kappa \lambda + 2 \varepsilon_{\mu \nu \lambda \kappa} \]

vanishes identically

\[ R_{\nu \mu \lambda \kappa} = 0 \]

Hence all the plastic imperfections are summarized in the torsion tensor in this case of the perfect tearing.

2. Representation of Moving Dislocations by Torsion Field

It will be shown that continuously distributed moving dislocations are represented by the torsion tensor field. All the components of the torsion tensor have their corresponding physical counterparts, one showing edge dislocations, another showing their motions, etc. We shall classify the types of dislocations and their motions by the components of the torsion tensor. Identities satisfied by these components will be derived, which can be compared with the equations of electromagnetic fields.

2.1. Classification of moving dislocations by the components of the torsion tensor. Since all imperfections are summarized in the torsion tensor in the theory, the curvature-like imperfections are emphasized. The physical meaning of the torsion must be clarified at first. When the torsion tensor \( S_{\mu \nu \lambda \kappa} \) exists, we cannot generally construct a parallelogram. Let us consider two vectors at \( A \), \( \overrightarrow{AB}(dx) \) and \( \overrightarrow{AC}(dx') \); and let \( \overrightarrow{CD} \parallel \overrightarrow{AB}, \overrightarrow{BE} \parallel \overrightarrow{AC} \) (see Fig. 3, where \( \overrightarrow{AB} \parallel \overrightarrow{CD} \) means that \( \overrightarrow{AB} = \overrightarrow{CD} \) and the lengths of \( \overrightarrow{AB} \) and \( \overrightarrow{CD} \) are equal). Generally, \( D \) and \( E \) do not coincide, and the discrepancy \( DE (ax') \) is expressed by

\[ \overrightarrow{AB} \text{ and } \overrightarrow{CD} \text{ are parallel and their lengths are equal.} \]
\[ \triangle x^\lambda = -S^{\lambda \mu \nu}_{\gamma \delta} \, \frac{dx^\mu}{2} \frac{dx^\nu}{2} \]

The physical roles of the torsion is embodied by dislocations such as shown in Fig. 4 and the discrepancy \( \triangle x^\lambda \) is called the Burgers vector. Hence one may call the torsion tensor the dislocation (density) tensor.

The types of dislocations and their motions can be classified into four classes by means of the types of the components of the torsion tensor. From (1.15), we have

\[ S_{\gamma \delta \tau}^{\lambda \kappa} = 0 \]

and

\[ S_{\gamma \tau \delta}^{\lambda \kappa} = -S_{\gamma \delta \tau}^{\lambda \kappa}. \quad (2.1) \]

Therefore, the different types of the components are represented by the following four

\[ S_{12}^{14}, \quad S_{12}^{33}, \quad S_{12}^{32}, \quad S_{12}^{13}. \]

The first two are the purely spatial components and the latter two are the mixed components. It is easy to see that the spatial components represent edge and screw dislocations.1)

i) \( S_{12}^{14} \): edge dislocation; the Burgers vector lying in the \( x' \)-direction, the dislocation line running along the \( x' \)-axis (which is perpendicular to the \( x'-x \) plane, if the axes are orthogonal) (Fig. 5),

ii) \( S_{12}^{33} \): screw dislocation; the Burgers vector lying in the \( x' \)-direction, the dislocation line running along the \( x' \)-axis (Fig. 6).

1) See [1], [3], [9], [11], etc.
Let us here consider the physical meaning of the mixed components of the torsion tensor. Considering the equation

$$\Delta x^\alpha = -\frac{1}{2} S_{\rho\sigma}^\alpha dx^\rho dx^\sigma dt,$$

we see that the material constituting a spatial vector AB represented by \((dx^\alpha, 0)\) is plastically changed by \(\Delta x^\alpha\) in a small time \(dt\) (see Fig. 7, where \(AB' \not\equiv AB\), \(AA' \not\equiv BB'\), and \(AA'\) and \(BB'\) are the world-lines of the material point A and B, respectively). Therefore, the component \(S_{\rho\sigma}^\alpha\) (Fig. 7) represents the decrement (or increment) of the number of the atoms contained in the element \(dx^\sigma\) after a small time \(dt\), is shown by the component \(S_{\rho\sigma}^\alpha\). This means the appearance of the vacancies (or interstitial masses) in the element, which is mainly caused by the climb or material non-conservative motion of dislocations.

On the other hand, \(S_{\rho\sigma}^\alpha\) shows that a part of the element constituting a vector \(AB\) (the atom B in Fig. 7) is plastically displaced in the \(x^\alpha\)-direction after a small time \(dt\), while the other part of the element (the atom A) is not. The plastic motion of this type is caused by the slip of dislocations. After the slip of a dislocation, the atoms above the slip surface are displaced in the direction of the Burgers vector, while the atoms under the surface are not (see Fig. 8).

We can examine the above meaning of \(S_{\rho\sigma}^\alpha\) from the projective viewpoint.\(^1\) It has been shown in D-XI [7] that \(S_{\rho\sigma}^\alpha\) means an edge dislocation and its motion along the

\(^1\) This point of view was suggested to the author by Prof. Kondo. See also D-VII[9].
direction of the Burgers vector is a slip comparable with the appearance of Orowan's pair. Now that we replace one of the lower indices, say 2, by 0, the time in which the slip takes place is measured by the length of glide along the direction of $x^1$

$$vdt = dx^1.$$  

This is the origin of $S^1_{02}$. From the slip motion of $S^1_{02}$, on the contrary, a climb of $S^1_{02}$ is expressed by $S^1_{02}$ in which the time of climb is measured by the length of climb along the direction of $x^1$. In a manner similar to this, we see that the slip of $S^1_{02}$ in the $x^3$-direction is also represented by $S^1_{02}$. These considerations are in exact correspondence with the projective viewpoint in Cartan's book [11], in which one of the fundamental points moves along the parameter line of $x^0$ (see Fig. 9).

In the theory of projective connexion, the torsion tensor is essentially different from the Riemann-Christoffel curvature tensor, both constituting parts of the Torsion-Curvature Tensor. Hence the relation between the motion of dislocations and the Riemann-Christoffel curvature tensor $\alpha$ as has been pointed out in b XI 7, will be reformulated in this projective formalism. Since the present theory is a simplified exposition within the framework of four-dimensional distant parallelism, the projective relation does not appear explicitly. This point will be clarified in a forthcoming Research Note.

From the above considerations, we can expect that the motion of a dislocation $S^1_{02}$ with velocity $v^1$ is represented by the mixed type component $S^1_{02}$ of the torsion tensor, satisfying

$$S^1_{02} = - S^1_{02} v^1. \quad (2.2)$$

This will be proved in § 2.2. From (2.2), we have for the mixed components
iii) $S^\ldot_{\mu\lambda} \dot{\nu}$, which includes the slip of dislocation; (appearance of Orowan's pair, etc.) It should be noted that two kinds of motions are represented by $S^\ldot_{\mu\lambda} \dot{\nu}$, one is the motion of $S^\cdot_{\mu\lambda} \dot{\nu}$ in the $x'$-direction, and the other the motion of $S^\cdot_{\mu\lambda} \nu$ in the $x'$-direction, the latter only meaning the appearance of Orowan's pair.

iv) $S^\cdot_{\mu\lambda} \dot{\nu}$, which means the climb of dislocations; (appearance of climb-wise pairs of edge dislocations or of interstitial masses etc.). Here it should be noted that always $S^\cdot_{\mu\lambda} \dot{\nu} = 0$, the motion of $S^\cdot_{\mu\lambda} \dot{\nu}$ in the $x'$-direction or the motion of $S^\cdot_{\mu\lambda} \nu$ in the $x'$-direction, only remaining.

2.2. Identities satisfied by the dislocation tensor. From the definition of the torsion or dislocation tensor, we have

$$S_{\mu\lambda} \dot{\nu} = - \partial_{\nu} \beta_{\mu\lambda},$$

(2.3)

where $\beta_{\mu\lambda} = v_{\mu\lambda}, \beta_{\nu} = f_{\nu} = 0$ and $f_{\mu\lambda}$ is the so-called asymmetric strain tensor. Splitting this equation into the spatial and the mixed components, we have

$$S_{\mu\lambda} \nu = - \partial_{\nu} \beta_{\mu\lambda},$$

$$S_{\mu\lambda} \nu = \frac{1}{2} \left( \partial_{\nu} v_{\mu\lambda} - \partial_{\nu} \beta_{\mu\lambda} \right).$$

(2.4)

This shows the relation between the torsion (dislocation and its motion), and the velocities of points and asymmetric strain tensor.

From (2.3), we have the identity

$$\partial_{\nu} S_{\mu\lambda} \dot{\nu},$$

(2.5)

or

$$\partial_{\nu} (\varepsilon_{\mu\nu} \dot{\nu} S_{\mu\lambda} \dot{\nu}) = 0,$$

which represents the non-divergent character of the dislocations.

---

1) $\varepsilon_{\mu\nu} \dot{\nu}$ is Eddington's $\varepsilon$, i.e., $\varepsilon_{\mu\nu} \dot{\nu} = -16$ an even (odd) permutation of $(0, 1, 2, 3)$, and is equal to 0, otherwise.
in the four-dimensional space. (2.5) splits into

\[
\begin{align*}
\partial_{\tilde{\nu}} S_{\tilde{\mu} \tilde{\lambda} \tilde{\kappa}} &= 0 \quad (\partial_{\tilde{\nu}} (\mathcal{E} \partial_{\tilde{\nu}} S_{\tilde{\mu} \tilde{\lambda} \tilde{\kappa}}) = 0), \\
\partial_{\tilde{\nu}} S_{\mu \lambda \kappa} &= 2 \partial_{\nu} S_{\mu \lambda \kappa}
\end{align*}
\]

(2.6)

The first equation of (2.6) means the non-divergent character of dislocations in the three-dimensional space. The latter equation relates the changes of dislocation distribution \( \partial_{\tilde{\nu}} S_{\tilde{\mu} \tilde{\lambda} \tilde{\kappa}} \) to the motion of dislocations represented by \( S_{\mu \lambda \kappa} \). Such has originated from the assumption of the four-dimensional distant parallelism criterion

\[ R_{\tilde{\mu} \tilde{\nu} \tilde{\lambda} \tilde{\kappa}} = 0. \]

**Theorem 1.** The motion of a dislocation represented by \( S_{\mu \lambda \kappa} \) moving with velocity \( v^\tau \) is represented by

\[ S_{\mu \lambda \kappa} = - v^\tau S_{\mu \lambda \kappa}. \]

**Proof.** The dislocation field moving with the velocity \( v^\tau \) is expressed as tensor functions of \( x^\tau - v^\tau t \),

\[ S_{\mu \lambda \kappa} (x^\tau - v^\tau t). \]

From the latter of (2.6), we have

\[ 2 \partial_{\nu} S_{\mu \lambda \kappa} = \partial_{\nu} S_{\mu \lambda \kappa} = - v^\tau \partial_{\nu} S_{\mu \lambda \kappa}. \]

Solving this equation, we have

\[ S_{\mu \lambda \kappa} = - v^\tau S_{\mu \lambda \kappa}. \]

**Example:** Let us here consider an example of a moving dislocation. An edge dislocation expressed by \( S_{\mu \lambda \kappa} \) is moving along the \( x^\prime \)-axis with velocity \( v \) (see Fig. 10). We have then the torsion...
tensor field such as follows:

\[
\begin{align*}
\sigma_{ij} &= \epsilon \delta(x' - vt) \delta(x^i - a) \epsilon - \sigma_{ij}' \\
\sigma_{\alpha \beta} &= -v \delta(x' - vt) \delta(x^\beta - a) \epsilon - \sigma_{\alpha \beta}'
\end{align*}
\]

all the other components are 0,

where \( \delta \) is Dirac's delta-function.

It has been discussed in Research Note No. 42 [11] as well as in a paper which will be published in R.A.E. Memoirs, III, that the electromagnetism is derived from the torsion tensor in a distant parallelism space of perfect tearing. Hence it is expected that \( \sigma_{ij}^{\pi} \) and \( \sigma_{\alpha \beta}^{\pi} \) can be compared with the electromagnetic fields. In fact, the first equation of (2.6) is compared with

\[ \text{div} \mathbf{H} = 0, \]

while the latter is with

\[ \text{rot} \mathbf{E} = \frac{\partial}{\partial t} \mathbf{H}, \]

where \( \epsilon^{\pi \alpha \beta} \sigma_{\alpha \beta}^{\pi} \) and \( \sigma_{\alpha \beta}^{\pi} \) are compared with the magnetic and dielectric fields, \( H^\pi \) and \( E^\pi \), respectively. 1)

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1) The other equations corresponding to other two Maxwell equations are given in D- [16]. See also [27].
incompatibility tensor, as well as its relation to the anelasticity tensor, will be clarified.

3.1. Incompatibility. When the deformation does not preserve the topology, i.e. some pairs of neighbouring atoms do not remain so after deformation, it is said to be plastic. In this case we cannot find displacement $u_\kappa$ of each point relative to a perfect lattice, but being required that $u_\kappa$ is related to the strain tensor $\varepsilon_{\lambda\kappa}$ by

$$\varepsilon_{\lambda\kappa} = \partial_\lambda u_\kappa. \quad \text{(3.1)}$$

The integrability condition for (3.1) is, as is well-known, vanishing of the curvature tensor $K_{\rho\mu\lambda}^{\kappa}$ by the Levi-Civita parallelism associated with the metric tensor $g_{\lambda\kappa}$, i.e.

$$K_{\rho\mu\lambda}^{\kappa} = 2 \sum_{\nu} \left\{ \frac{\kappa}{\rho \mu \lambda} + 2 \left\{ \frac{\nu}{\nu \mu \lambda} \right\} \right\} = 0. \quad \text{(3.2)}$$

In the case of plastic deformation, (3.2) does not hold, and the non-zero tensor $K_{\rho\mu\lambda}^{\kappa}$ is called the incompatibility tensor.

The spatial components $K_{\rho\mu\lambda}^{\kappa}$ of the curvature or incompatibility tensor is the ordinary incompatibility which means the existence of residual strains (stresses). On the other hand, the mixed components such as $K_{\rho\mu}^{\kappa}$ and $K_{\rho\lambda}^{\kappa}$ represent that the plastic deformation is now going on. Let us investigate the character of the four-dimensional incompatibility tensor.

Bianchi's identity can be written as

$$\partial_\rho K_{\nu\rho\lambda}^{\kappa} = 0,$$

$$\begin{array}{l}
\partial_\rho (\varepsilon_{\sigma\rho\nu} K_{\nu\rho\lambda}^{\kappa}) = 0 \end{array} \quad \text{(3.3)}$$

which shows the non-divergent character of the incompatibility. The spatial part of (3.3) is rewritten as

1) See [4].
\[ \partial_t \tilde{K}_{\overline{\rho} \overline{\sigma} \overline{\lambda}} = 0, \]
\[ (\partial_t (\varepsilon^{\overline{\rho} \overline{\sigma} \overline{\lambda}} K_{\overline{\rho} \overline{\sigma} \overline{\lambda}}) = 0 ). \]

The mixed components are rewritten as
\[ \partial_t \tilde{K}_{\overline{\rho} \overline{\sigma} \overline{\lambda}} = 2 \partial_t \tilde{K}_{\overline{\rho} \overline{\sigma} \overline{\lambda}}, \]
\[ \partial_t \tilde{K}_{\overline{\rho} \overline{\sigma} \overline{\lambda}} = 2 \partial_t \tilde{K}_{\overline{\rho} \overline{\sigma} \overline{\lambda}}. \]

Since (3.5) and (3.6) can be transformed into
\[ \varepsilon^{\overline{\rho} \overline{\sigma} \overline{\lambda}} \partial_t \tilde{K}_{\overline{\rho} \overline{\sigma} \overline{\lambda}} = 2 \partial_t (\varepsilon^{\overline{\rho} \overline{\sigma} \overline{\lambda}} K_{\overline{\rho} \overline{\sigma} \overline{\lambda}}), \]
\[ \varepsilon^{\overline{\rho} \overline{\sigma} \overline{\lambda}} \partial_t \tilde{K}_{\overline{\rho} \overline{\sigma} \overline{\lambda}} = 2 \partial_t (\varepsilon^{\overline{\rho} \overline{\sigma} \overline{\lambda}} K_{\overline{\rho} \overline{\sigma} \overline{\lambda}}), \]
respectively, \( K_{\overline{\rho} \overline{\sigma} \overline{\lambda}} \) can be interpreted as the flow of the spatial incompatibility, and \( K_{\overline{\rho} \overline{\sigma} \overline{\lambda}} \) can be as the flow of \( \tilde{K}_{\overline{\rho} \overline{\sigma} \overline{\lambda}} = 0 \). Hence we have

**Theorem 2.** The flow of the spatial incompatibility is represented by \( K_{\overline{\rho} \overline{\sigma} \overline{\lambda}} \) for which the equation of continuity holds,
\[ \partial_t (\varepsilon^{\overline{\rho} \overline{\sigma} \overline{\lambda}} K_{\overline{\rho} \overline{\sigma} \overline{\lambda}}) = 2 \partial_t (\varepsilon^{\overline{\rho} \overline{\sigma} \overline{\lambda}} K_{\overline{\rho} \overline{\sigma} \overline{\lambda}}). \]

Since the incompatibility is changed by moving dislocations they are mutually related. Using a formula in Schouten's book (4.22b, p141 in [14]) and taking account of vanishing of the Riemann-Christoffel curvature tensor, we have
\[ K_{\overline{\rho} \overline{\sigma} \overline{\lambda} \mu} = 2 \partial_{\overline{\rho}} S_{\overline{\rho} \mu \overline{\sigma} \overline{\lambda}} + 2 \partial_{\overline{\sigma}} S_{\overline{\sigma} \mu \overline{\rho} \overline{\lambda}}. \]

Splitting this equation into the spatial and the mixed parts, it holds that
\[ K_{\overline{\rho} \overline{\sigma} \overline{\lambda} \mu} = 2 \partial_{\overline{\rho}} S_{\overline{\rho} \mu \overline{\sigma} \overline{\lambda}} + 2 \partial_{\overline{\sigma}} S_{\overline{\sigma} \mu \overline{\rho} \overline{\lambda}}; \]
\[ K_{\overline{\rho} \overline{\sigma} \overline{\lambda} \mu} = \partial_{\overline{\sigma}} S_{\overline{\sigma} \mu \overline{\rho} \overline{\lambda}} + 2 \partial_{\overline{\lambda}} S_{\overline{\lambda} \mu \overline{\rho} \overline{\sigma}}; \]
\[ K_{\overline{\rho} \overline{\sigma} \overline{\lambda} \mu} = 2 \partial_{\overline{\lambda}} S_{\overline{\lambda} \mu \overline{\rho} \overline{\sigma}}. \]

1) Note that \( K_{\overline{\rho} \overline{\sigma} \overline{\lambda} \mu} = 0 \) holds.
3.2, Anelasticity tensor. The change of the plastic state is represented by the change of the images of the mapping in the natural state. Let a spatial vector \( s^\alpha \) be mapped to \( s^\alpha \) in the natural state, and after a time \( dt \) let the same material be mapped to \( s^\alpha + ds^\alpha \). Since there is no distortion in the natural state, we can conclude that some plastic deformation occurs in the material constituting the vector \( s^\alpha \). The plastic change \( \Delta s^\alpha \), or \( \Delta s^\alpha = B^\alpha_\beta \Delta s^\beta \), can be expressed by the torsion tensor as

\[
\Delta s^\alpha = -\frac{1}{2} S^\alpha_{\beta \gamma} s^\beta ds^\gamma.
\] (3.9)

Hence we see that the mixed components of the torsion tensor \( S^\alpha_{\beta \gamma} \) represent the time rate of purely plastic deformation in agreement with § 2,1) The symmetric part \( S_{\alpha (\beta \gamma)} \) denotes the plastic deformation and the antisymmetric part \( S_{\alpha \beta \gamma} \) denotes the plastic rotation, each of which corresponds to the deformation and the rotation of the image in the natural state. Since the natural length of a material is measured by the length in its natural state, the length is changed by the plastic deformation (the elastic deformation does not affect it). The rate of the change will be obtained using the symmetric part \( S_{\alpha (\beta \gamma)} \) as follows,

\[
\frac{D}{D t} ds^\alpha = 4 S_{\alpha (\beta \gamma)} dx^\beta dx^\gamma.
\] (3.10)

where \( ds^\alpha = g_{\alpha \beta} dx^\beta dx^\alpha \).

Eckart defines the anelasticity tensor \( a_{\alpha \beta} \) [10]

by

\[
\frac{D}{D t} ds^\alpha = 2 a_{\alpha \beta} dx^\beta dx^\alpha.
\] (3.11)

By calculating

\[
D ds^\gamma = (g_{\lambda \alpha} + \frac{\partial g_{\lambda \alpha}}{\partial t} \frac{dt}{t}) dx^\lambda dx^\alpha - g_{\lambda \alpha} dx^\gamma dx^\lambda dx^\alpha
\]

\[
= \left[ \frac{\partial g_{\lambda \alpha}}{\partial t} + 2 \varepsilon_{\lambda \alpha \beta} \right] dx^\lambda dx^\alpha dt,
\]

1) It should be noted that elastic deformation does not affect the natural state.

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where
\[ \frac{d\bar{x}}{dt} = d\bar{x} + \frac{\partial}{\partial \bar{x}^p} d\bar{x}^p dt, \]
and higher order terms are neglected here, we find that the anelasticity tensor is exactly the same as the mixed component \(2S_{o(\bar{x}, \bar{\bar{x}})}\) of the torsion tensor.

**Theorem 3.** The anelasticity tensor, by which the time rate of the change of the natural metric tensor is represented, is the twice of the symmetric part of the mixed component \(S_{o(\bar{x}, \bar{\bar{x}})}\) of the torsion tensor,

\[ a_{\bar{x}\bar{\bar{x}}} = 2S_{o(\bar{x}, \bar{\bar{x}})}. \tag{3.12} \]

**Proof.**
\[ a_{\bar{x}\bar{\bar{x}}} = \frac{1}{2} \left\{ \frac{\partial}{\partial t} g_{\bar{x}\bar{x}} + 2(\partial_{\bar{x}} v_{\bar{x}}) \right\} = \partial_{\bar{x}} v_{\bar{x}} - \partial_{\bar{x}} E_{\bar{x}\bar{x}} \]
\[ = \partial_{\bar{x}} v_{\bar{x}} - \partial_{\bar{x}} f(\bar{x}, \bar{\bar{x}}) = 2S_{o(\bar{x}, \bar{\bar{x}})}. \]

The equation connecting \(a_{\bar{x}\bar{\bar{x}}}\) with the dislocation field is obtained from (2.6)
\[ 4 \partial_{\bar{x}} \partial_{\bar{x}} a_{\bar{x}\bar{\bar{x}}} = 2 \partial_{\bar{x}} (\partial_{\bar{x}} S_{(\bar{x}, \bar{\bar{x}})}) + \partial_{\bar{x}} S_{(\bar{x}, \bar{\bar{x}})} \bar{a}_{\bar{x}}. \tag{3.13} \]

Using (3.8), we obtain the relation between the incompatibility and the anelasticity tensor
\[ K_{o\bar{x}\bar{\bar{x}}} = \partial_{\bar{x}} a_{\bar{x}\bar{\bar{x}}}, \]
\[ \partial_{\bar{x}} K_{o\bar{x}\bar{\bar{x}}} = 4 \partial_{\bar{x}} \partial_{\bar{x}} a_{\bar{x}\bar{\bar{x}}}. \tag{3.14} \]

Hence, we have

**Theorem 4.** The mixed components \(K_{o\bar{x}\bar{\bar{x}}}\) of the curvature tensor of the metric tensor \(g_{\bar{x}\bar{x}}\) mean the rate of the change of the anelasticity tensor.
It is interesting to see that the rate of the change of the spatial incompatibility of the strain is exactly the Riemann-Christoffel curvature tensor of the space having the anelasticity tensor as the Riemannian metric. In other words, we can define a Riemannian space which is the time derivative of the strain metric. It shows that the plastic deformation does not necessarily change the incompatibility $\kappa_{\vec{x}_i}$. Only such plastic deformations, whose anelasticity tensor is Riemannian (i.e., not Euclidean) or the deformation by which the image in the natural state is incompatibly changed, produces the spatial incompatibility.

It should also be noted that the anti-symmetric part $S_{\alpha \beta \gamma \delta}$ of the mixed part of the torsion tensor does not affect the natural metric. In other words, the plastic deformation such as denoted by $S_{\alpha \beta \gamma}$ induces neither the metrical imperfection nor the incompatibility.

4. Discussions

i) It seems to be but a step to extend this distant parallelism theory to a general non-Riemannian plasticity theory including moving curvature-like imperfections as well as torsion-like ones. We can proceed along the lines of D-IX [11]. Moreover, a four-dimensional non-Riemannian space can be imbedded in a Euclidean (or Minkowskian) space of higher dimensionality, as has been treated in the elegant theory of Kondo. What rôle is played by the Euler-Schouten curvature tensor, which is pointed out to be more fundamental than the Riemann-Christoffel curvature tensor [5], is the next problem to be investigated.

ii) The projective viewpoint has been suggested by Kondo in D-VII [9]. Both the curvature and the torsion tensor are combined into one geometrical concept, and the

1) See, e.g., [41, 51, 5].
relation between the Riemann-Christoffel curvature tensor and the motion of dislocations shown in D-XI [7] will be clarified, if our point of view is extended to the projective one.

iii) It is not so difficult to obtain the stress and strain when a moving dislocation field is given. If an extended four-dimensional stress tensor \( \sigma^{\lambda\mu} \) is defined by

\[
\sigma^{\lambda\mu} = E^{\lambda\mu\rho\sigma} \varepsilon_{\rho\sigma},
\]

where \( E^{\lambda\mu\rho\sigma} \) is the ordinary elastic modulus tensor, \( E^{\lambda\rho\kappa\nu} = \rho \delta^{\lambda\nu} \), \( \rho \) is the density of the material, and the other independent components vanish, the wave equation is represented by

\[
\partial_{\lambda} \sigma^{\lambda\mu} = 0.
\]

Since the curvature tensor calculated from the metric tensor \( g_{\lambda\mu} \) is, on account of vanishing of the Riemann-Christoffel curvature tensor, related by the torsion tensor as follows

\[
K_{\rho\kappa\lambda} = 2 \partial_{(\rho} S_{\lambda\kappa\mu)} + 2 \partial_{(\rho} S_{\mu\kappa\lambda)}
\]

we can obtain the stress and the strain by solving (4.1), (4.2) and (4.3). Schaefer's stress function tensor will be extended to the four-dimensional case.

iv) Considering from the microscopic viewpoint, dislocations exist as dislocation lines in the three-dimensional space. They constitute a 1-dimensional network in which the conservation law of Burgers vectors in analogy with Kirchhoff's law holds.

Since the trace of a dislocation line forms a surface in the four-dimensional space, it will be expected that the dislocations constitute a 2-dimensional network in the four-dimensional space. The conservation law of Burgers vectors will also hold by virtue of the

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1) See [16], [18].

4) See [9].
non-divergent character of the torsion field. The career of a dislocation, e.g. the birth, growth, death, etc., will completely be represented by the 2-dimensional dislocation network.
References


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