Linear Programming with Weak Graphical Representation

By

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INTRODUCTION

During the past seven years of the applied-geometrical researches pushed forth by the RAAG members with abundant heritage from the Unifying Study Group of pioneer Japanese tensorists, topological network theory was thoroughly studied to the establishment of its rigid foundation together with many practical applications [1]. Starting from linear electric network theory, unoriented and oriented switching circuits were attacked, and, furthermore, transportation networks were subjected to the topological investigation. According to the development of the study, the algebraic systems to be under consideration have been extended from additive groups to lattices, and then to more general ones. With the aim of establishing a theory of the most general kind of networks, the theory of general information networks was recently proposed by one of the present authors [2], [3], where it was also pointed out that the topological theory of transportation networks, such as presented in [4], was one of the best examples of the general theory.

From the standpoint of Operations Research, the problems of transportation networks are a special case of the problems of Linear Programming, where the variables to be optimized satisfy some linear constraints (usually called "Kirchhoff's current law" in electric network theory) determined by the topological structure of the underlying network. There are, however, found also the problems of linear-programming type which are more general than those of transportation networks but are subject to the constraints not entirely irrelevant to the network structure. We shall call, in this Note throughout, such problems (whose precise definition will be found in §2) problems of "Linear Programming with Weak Graphical Representation" in contrast with the problems of transportation networks to be called problems of "Linear Programming with Strong Graphical Representation". The research to be reported in this Note was started when one of the present authors discovered the fact that a problem 1) which had been believed to be unsolvable by the usual

method for solving transportation problems (such as the stepping-stone method) but to be solved only by the general simplex method, could be solved by entirely the same method as that proposed in [2] and [4]. Thenceforth, the authors were discussing the possibility of establishing the theory of linear programming with weak graphical representation in the framework of the theory of general information networks. Although the problem has not yet been solved satisfactorily enough, the authors dare to report, in this Note, the results they have attained so far, expecting further discussions from a wider circle.

One of the characteristics of the problems of linear programming with weak graphical representation is the weakness of its dependence on the underlying network structure. In a problem of transportation network the variables satisfy the continuity condition (so-called Kirchhoff's current law) determined by the structure of the network, while in a problem of linear programming with weak graphical representation the network structure does not produce such a quantitative constraint but imposes only a qualitative (as it were) restriction on the variables. In that sense the network theory developed in this Note may be regarded as a graphical theory of computer-programming because the network structure is connected with the process of computation, but not with the computed values themselves.

For our general attitude on network problems as well as for the detailed explanation of symbolism regarding network topology, the readers may refer to [2], [3] and [4] as well as to [6].
A Simple Example

In order to make the general definition in the following section §2 plausible let us begin with a simple example of linear programming with weak graphical representation. This example will be to the general problem to be dealt with in this Note what the Hitchcock problem is to the problem of transportation networks of the general kind.

Suppose that three machines are available for the production of six kinds of parts, and that the numbers of parts to be produced in a month and the machine-hours available in a month, as well as the production hours required per unit amount of part per machine, are known. If the production cost is proportional to the total machine time, what is the best production schedule to minimize the production cost? The data concerning this problem are as shown in Table 1.

<table>
<thead>
<tr>
<th>Machine j</th>
<th>Part i</th>
<th>Production hours per unit per machine ( (e_{ij}) )</th>
<th>Number of parts required ( (a_i) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>3 4 2</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3 1 2</td>
<td>40</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>2 1 5</td>
<td>60</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>5 2 1</td>
<td>20</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>2 2 1</td>
<td>20</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>1 1 2</td>
<td>30</td>
</tr>
</tbody>
</table>

Available machine hours \( b_j \) = 80 30 160 = 270

1) We take up the same example as the one cited in the footnote 1) on p.1. The solution to this problem will be shown in §10, Ex.3. See also §9,3.
The problem is formulated in the form of linear programming, by introducing the variables \( x_{ij} \)'s which indicate the number of part \( i \) produced by machine \( j \), as follows.

\[
\text{Minimize} \quad z = \sum_{i=1}^{n} \sum_{j=1}^{m} e_{ij}x_{ij},
\]

under the conditions:

\[
\sum_{j=1}^{m} x_{ij} = a_{i} \quad (i=1, 2, 3, 4, 5, 6), \tag{1.2} \]

\[
\sum_{i=1}^{n} e_{ij}x_{ij} \leq b_{j} \quad (j=1, 2, 3), \tag{1.3}
\]

and

\[
x_{ij} \geq 0 \quad (i=1, 2, 3; j=1, 2, 3, 4, 5, 6). \tag{1.4}
\]

(1.2)

(1.3)

(1.4)

(It will readily be noticed that, if all the \( e_{ij} \)'s in (1.3) were unity, the problem would be the same as a Hitchcock transportation problem.)

In the same way as in [4], a network model can be constructed for this problem. In Fig. 1, the branches connecting nodes \( i=1, \ldots, 6 \) and nodes \( j=1, 2, 3 \) convey the

![Fig. 1 Network Model](image-url)

---

1) \( \sum_{j=1}^{m} x_{ij} \geq a_{i} \) is the exact interpretation of the problem. But it will not be difficult to see that (1.2) is equivalent to that condition under the present circumstances.
flows of the values equal to $x_{ij}$'s, which we shall call "currents" for convenience' sake; $e_{ij}$'s are represented by the voltage sources$^1$; the black boxes are placed in order to realize the restrictions to be imposed on currents, i.e. they have the voltage-current relation shown in Fig. 2; the continuity of current, namely Kirchhoff's current law, is assumed to hold at every node; and finally, the elements (to be called "transformers") which have the characteristics shown in Fig. 3 are inserted in the vicinity of the nodes labelled by $j$'s. Moreover, let us consider the problem of maximizing the input current, in the network of Fig. 1, in such a manner that the continuity of current as well as that of voltage may hold and that the configuration of current and voltage may satisfy all the characteristics of the branches (such as shown in Figs. 2 and 3). Then it will be obvious that the currents in the intermediate branches (namely, those between nodes $i$ and $j$) satisfy the conditions (1.2), (1.3) and (1.4).$^2$ Furthermore, we shall be

---

1) An alternative position at which a voltage source may be put will be shown in Fig. 13 of Example ($§10$). The network representation of Fig. 13 is easier to treat.

2) So long as there exists any solution at all to the original problem.
convinced that those currents constitute, indeed, an optimum solution to the original problem, if we take account of the fact that the dual problem is

\[ w = \sum_{i} a_i x_i - \sum_{j} b_j \beta_j \]  \hspace{1cm} (1.5)

under the conditions:

\[ e_{ij} \geq x_i - e_{ij} \beta_j \] for all \( i \) and \( j \), \hspace{1cm} (1.6)

\[ \beta_j \geq 0 \] for all \( j \),

the minimax relation is

\[ z = \sum_{i} \sum_{j} e_{ij} x_{ij} \geq \sum_{i} \sum_{j} (x_i - e_{ij} \beta_j) x_{ij} \]

\[ = \sum_{i} x_i \sum_{j} x_{ij} - \sum_{j} \sum_{i} e_{ij} x_{ij} \]

\[ \geq \sum_{i} a_i x_i - \sum_{j} b_j \beta_j = w, \] \hspace{1cm} (1.7)

hence

\[ \min z \geq \max w, \] \hspace{1cm} (1.8)

(1.2)~(1.4) (1.6)

and the optimality conditions (i.e., the conditions for the inequality in (1.7) to be reduced to an equality) are

\[ e_{ij} = x_i - e_{ij} \beta_j \text{ if } x_{ij} > 0 \] \hspace{1cm} (1.9)

\[ \beta_j = 0 \text{ if } \sum_i e_{ij} x_{ij} < b_j, \]

where the dual variables \( x_i \) and \( \beta_j \) may be regarded as the analogues of the voltages at nodes \( i \) and \( j \), respectively.

The above-explained example, which can be compared to the Hitchcock problem in the case of transportation problems, are naturally extended to the general problem to be treated in the following sections.
2. Definition of the Problem

To begin with, let us consider the following problem. Given constants\(^1\):

\[
\mathcal{A}^\kappa \quad (\geq -\infty, < \infty; \ k = 1, 2, \ldots, m; \ \kappa = 1, 2, \ldots, n);
\]

\[
c^\kappa \quad (\geq 0, \leq \infty; \ \kappa = 1, 2, \ldots, n);
\]

\[
e^\kappa \quad (\geq 0, < \infty; \ \kappa = 1, 2, \ldots, n).
\]

Variables\(^1\):

\[
\mathcal{B}^\kappa \quad (\geq 0, < \infty; \ \kappa = 1, 2, \ldots, n);
\]

\[
u_a \quad (\geq -\infty, \leq \infty; \ a = 1, 2, \ldots, m);
\]

\[
E^\kappa \quad (\geq 0, \leq \infty; \ \kappa = 1, 2, \ldots, n);
\]

\[
E^{\kappa'} \quad (\geq 0, \leq \infty; \ \kappa = 1, 2, \ldots, n).
\]

Problem:

To determine the relation between \(u_1\) (one of the variables introduced above) and the \(\mathcal{B}\) defined by (2.1):

\[
\mathcal{B} = \bigoplus \mathcal{B}^\kappa
\]

(2.1)\(^2\)

as well as the corresponding values of \(\mathcal{B}^\kappa\)'s and \(u_a\)'s, under the conditions:

\[
\mathcal{B}^a = 0 \quad \text{for} \quad a = 2, 3, \ldots, m-2, m-1,
\]

\[
0 \leq \mathcal{B} \leq c^\kappa \quad \text{for} \quad \kappa = 1, 2, \ldots, n;
\]

\(u_m = 0,\)

\[
E^\kappa \cdot E^{\kappa'} = 0 \quad \text{for} \quad \kappa = 1, 2, \ldots, n,
\]

\[
\mathcal{B}^a u_a - (E^\kappa - E^{\kappa'}) = e^\kappa \quad \text{for} \quad \kappa = 1, 2, \ldots, n;
\]

1) The electric network analogy of these constants and variables will be shown later in §4.

2) Throughout this note, Einstein's summation convention will be used without special remark.
\[ \begin{align*}
S^x &= c^x \quad \text{if} \quad E_x > 0, \\
\lambda^x &= 0 \quad \text{if} \quad E' > 0.
\end{align*} \] (2.4)

If \( D^x \) is very general, we have a general linear-programming problem. The problem we shall study in the sequel is the special case of the above general problem in which

\[ D^x = \text{sgn} L^x \] has a graphical representation,\(^1\)

i.e., there exists a linear graph consisting of \( n \) branches \( \tau^x \)'s and \( m \) nodes \( \tau^z \)'s such that the incidence relation between the branches and the nodes is expressed by the matrix \( D^x = \text{sgn} L^x \):

\[ \tau^y D^x = \tau^x \tau^z. \] (2.5)

where \( \text{sgn} \) indicates, as usual, the function defined by (2.6):

\[ \text{sgn } x = \begin{cases} 
-1 & (x < 0), \\
0 & (x = 0), \\
1 & (x > 0).
\end{cases} \] (2.6)

In order to avoid trivial discussions, we assume the representative graph as connected. A transportation-network problem, investigated in [4], is a special case in which \( L^x = \text{sgn} L^x \), i.e., \( L^x \) itself has a graphical representation. Thus, the constants \( L^z \)'s are completely determined by the incidence relation of the corresponding graph in a transportation-network problem (hence we call it a problem of "linear programming with strong graphical representation), while, in a problem of "linear programming with weak graphical representation, only part of it is reflected on the geometrical structure of the corresponding graph.

We define, furthermore, two associated quantities, namely

\[ z = e^x S^x \quad \text{and} \quad w = S^y - c^x E_x, \] (2.7)

---

1) For the sake of simplicity, we assume that no graph contains a branch starting from a node and ending at the same node. (If such is contained, we may replace it two branches connected in series, introducing a new node.) But this assumption is not essential in the following treatment of ours. cf. [2] and [3].
for the purpose of later use, introducing, in addition, the
notation defined by
\[
 z(\beta_0) = \min_{\beta} z \quad (2.8)
\]
\[
 \beta = \beta_0 \text{(const.)}
\]

The fact that \( z(\beta_0) \) is a well-defined single-valued
continuous function\(^1 \) of \( \beta_0 \) will be so obvious that we may
omit its proof.
3. Significance of the Problem

Suppose the problem defined in §2 has been solved. Then the solutions of the following problems can readily be obtained.

(i) To minimize \( z = e_x S^x \) under the conditions (2.2) and \( S = \text{const.} \) (given): — The set of \( S^x \)'s, obtained for the problem in §2, corresponding to the given value of \( S \) is the optimum solution for this problem, because, for any set of \( S^x \)'s satisfying (2.2) and \( S = \text{const.} \) and any set of \( u_a \)'s, \( E_x \)'s and \( E'_x \)'s satisfying (2.3), we have

\[
z = e_x S^x = (S^x u_a - E_x + E'_x)S^x = S u_1 - E_x S^x + E'_x S^x
\]

hence

\[
\min_{(2.2)} \quad \max_{S=\text{const.}} z \geq \max_{(2.3)} w, \tag{3.1}
\]

and the condition (2.4) reduces the inequality in (3.1) to an equality, thus making the corresponding \( S^x \)'s the optimum solution.

(ii) To maximize \( S \) under the condition (2.2) and \( z = e_x S^x < \text{const.} \) (given): — As will be shown later, \( \min_{(2.2)} z \) is a monotonously non-decreasing function of \( S \).

\[
\min_{(2.2)} \quad \max_{\text{const.}} S
\]

Therefore an \( S \) is known to be the maximum of all the \( S \)'s realizable for a fixed \( z \) if the \( z \) is minimum for that \( S \).

The problem taken as an example in §1 is obviously of the type (i). We can consider also a problem which has the same graphical representation as the problem in §1 but is more general than it, i.e. the problem of

Minimizing \[ z = \sum_i \sum_j e_{ij} x_{ij}, \tag{3.3} \]

under the conditions:

\[
\sum_i d_{ij} x_{ij} = a_i, \quad \sum_j d_{ij} x_{ij} \leq b_j, \quad x_{ij} \geq 0.
\]

The problem mentioned on p. 371 - p. 376 of [5] as practically unsolvable belongs to this type.
4. Electric Model of the Problem

It is in order to make the following descriptions concise, but not in order to make use of any results known in the theory of electric networks, that we introduce the electric-circuit model in this section.

First we define a network whose topological structure is determined by the incidence matrix \( \mathbf{D} \) and \( \mathbf{J} \); i.e., if we denote the branches and the nodes of the network by \( J_{\alpha} (\alpha = 1, 2, \cdots, n) \) and \( \mathcal{J} ^{a} (a = 1, 2, \cdots, m) \), respectively, the boundary operator \( \mathcal{E} \) is expressed as

\[
\mathcal{E} \mathcal{J} ^{a} = \mathbf{D}^{a} \mathbf{J} ^{a} .
\]  

(4.1)

As regards variables, we make \( \mathcal{B}^{\alpha} \) correspond to the current in branch \( \mathcal{J} ^{\alpha} \) and \( u_{a} \) to the voltage at node \( \mathcal{J} ^{a} \). In view of the first condition of (2.3), the m-th node, or \( \mathcal{J} ^{m} \), is assumed to be grounded. (For \( E_{\alpha}'s \) and \( E_{\alpha}'s \), see below.) Then we specify the structure of each branch as shown in Fig. 4, where \( \varepsilon e_{\alpha} \) is the voltage source with e.m.f. \( e_{\alpha} \), \( \varepsilon e_{\alpha} \) is an element, like an ideal transformer, which has the characteristic defined in Fig. 5, and the black box

![Fig. 4](image)

![Fig. 5](image)
has the current-voltage relation shown in Fig.6. The variables \( E_\infty \) and \( E'_\infty \) are thus put in analogy with the voltage across the black box. We define, furthermore, the "state" of a branch, by the current in and the voltage across the black box contained in the branch, as indicated by \( A, B, C, D \) and \( E \) in Fig.6.

The final form of the network will be such as shown in Fig.7. The first condition of (2.2) is, in terms of the electric model circuit, satisfied by virtue of the continuity of current (Kirchhoff's current law) for the nodes \( \gamma_s, \ldots, \gamma_m \), the \( S \) in (2.1) corresponds to the current flowing into the network through node \( \gamma_s \) (hence we shall call the "input node") from outside, and no restriction is imposed on the value of \( S^x \) because \( \gamma_m \) (to be called the "output node") is grounded. The second condition of (2.2) is obviously satisfied by virtue of the characteristic of the black box (see Fig.6). The first condition of (2.3) has already been mentioned above. The second of (2.3) is merely the condition introduced for the purpose of expressing an arbitrary quantity in the form of the difference \( (E_\infty - E'_\infty) \) of two non-negative quantities in a unique way. The third of (2.3) states the fact that the potential difference between the two terminals of a branch is equal to the sum of the voltages across the component elements which, connected in series, constitute the branch. The condition (2.4) is again satisfied by virtue of the characteristic of the black box (Fig.6).

Thus, the problem defined in §2 is restated in electrical terminology as that of determining the relation between the current flowing into the network of Fig.7 through the input node and the voltage at the input node, as well as the corresponding current and voltage configurations. The associated cost function \( z = e_\infty S^x \) obviously corresponds to the total power injected into the voltage sources.

---
1) Note that the amount of the current flowing out of the network through \( \gamma_m \) into the ground is not necessarily equal to that of the input current.
5. Outline of the Solution

The basic principle of solution is entirely the same as that we based ourselves upon in order to solve a transportation network (see (4)). We repeat the current-increasing steps and the voltage-increasing steps alternately. This repetition terminates with a finite number of steps. The relation between \( S \) and \( \mu_1 \) is monotonously non-decreasing, and it suffices to consider only the case \( S \geq 0 \) because the case \( S \leq 0 \) will be treated in entirely the same manner (or, speaking more precisely, by inverting the sign of \( S \) as well as that of \( u_\alpha \)'s and \( E_x \)'s).

5.1. Current-increasing steps. Let us suppose that a pair of a current and a voltage configuration (namely, a set of variables \( S_x \)'s, \( u_\alpha \)'s, \( E_x \)'s and \( E'_x \)'s) is given, together with \( S \) and \( z = e_x S_x \), which satisfies (2.1) - (2.4). As has already been proved in §3,

\[
\begin{align*}
z &= z(S) \quad (5.1)
\end{align*}
\]

for such a current configuration. We then require to increase the value of \( S \) as much as possible with all the voltages fixed.

If we denote the given currents by \( S \) and \( S_x \)'s and the incremental ones by \( \Delta S \) and \( \Delta S_x \)'s, then the conditions for the resulting currents \( 'S = S + \Delta S' \) and \( 'S_x = S_x + \Delta S_x \) to satisfy (2.1) - (2.4) in regard to the given voltages \( u_\alpha \)'s, \( E_x \)'s, and \( E'_x \)'s are expressed as follows,

\[
\begin{align*}
\Delta \lambda_x &= \delta_x \Delta \lambda^x \\
\delta_x \Delta S_x &= 0 \quad \text{for} \quad a=2,3,\ldots,m-1, \\
-\lambda^x &\leq \Delta \lambda_x \leq \lambda^x \quad \text{for all} \quad x, \\
\Delta \lambda^0 &= 0 \quad \text{either} \quad E_x \quad \text{or} \quad E'_x > 0.
\end{align*}
\]

(5.2) states that the continuity condition of the same form as (2.1) and (2.2) holds also for the incremental currents, and that the branches in state \( A \) or \( B \) with respect to the given current and voltage configuration are non-conductive for the incremental currents, those in state \( E \) are...
conductive only in the negative direction with capacity \(-S^x = -c^x\), those in state \(\mathcal{C}\) are conductive both in the positive and in the negative direction with capacities \(c^x - S^x\) and \(-S^x\) in the respective directions, and those in state \(\mathcal{D}\) are conductive only in the positive direction with capacity \(c^x\).

With these conditions for incremental currents we search for a conduction route starting from the input node \(\sigma^0\). By a conduction route starting from \(\sigma^0\) we mean such a route as shown in Fig. 8. Before making the specification of a conduction route in Fig. 8 we should first define the term "route". A "route" (of length \(l\)) is defined as a sequence of nodes and (directed) branches

\[
\{\sigma_{x_1}, \xi, \mathcal{C}_{x_{i+1}}, \mathcal{T}_{x_{i}}, \mathcal{D}_{x_{i+1}}, \mathcal{D}_{x_{i}}, \ldots, \mathcal{D}_{x_{j}}, \xi, \mathcal{T}_{x_{j}}, \mathcal{D}_{x_j}, \ldots, \sigma_{x_n}, \xi, \mathcal{T}_{x_{n}}, \mathcal{D}_{x_n}\}
\]

such that

\[
\xi_i \in T_{x_i} = \sigma_{x_{i-1}} - \sigma_{x_i} \quad (i=1,2,\ldots,l),
\]

where

\[
\xi_i = +1 \quad \text{or} \quad -1.
\]

(5.3)

(5.4)

A branch \(\mathcal{T}_{x_i}\) belonging to the sequence with \(\xi_i = +1\)(or \(-1\)) is said to be "contained in the route with positive (or negative) sign". The conduction route of the type shown in Fig. 8(a) is a route for which

\[
\begin{align*}
\sigma_{a_n}^0 &= \sigma_1^0 \quad \text{(input)}, \\
\sigma_{b_n}^0 &= \sigma_m^0 \quad \text{(output),}
\end{align*}
\]

\[
\sigma_n^0 \neq \sigma_j^0 \quad \text{if} \quad i \neq j,
\]

and every branch contained with positive (negative) sign is conductive in the positive (negative) direction. The branch currents along the route are determined successively (if the current entering the input node \(\sigma^0\) is \(\Delta S > 0\)):

\[
\begin{align*}
\xi_1 \, \Delta S^x &= \left| \frac{1}{\mathcal{B}_{x_i}^0} \right| \Delta \lambda, \\
\xi_2 \, \Delta S^x &= \left| \frac{\mathcal{B}_{x_i}^0}{\mathcal{B}_{x_i}^0} \right| \Delta \lambda, \\
\xi_i \, \Delta S^x &= \left| \frac{\mathcal{B}_{x_i}^0 \mathcal{B}_{x_i}^0 \ldots \mathcal{B}_{x_i}^0}{\mathcal{B}_{x_i}^0 \mathcal{B}_{x_i}^0 \ldots \mathcal{B}_{x_i}^0} \right| \Delta \lambda, \\
\xi_n \, \Delta S^x &= \left| \frac{\mathcal{B}_{x_n}^0 \mathcal{B}_{x_n}^0 \ldots \mathcal{B}_{x_n}^0}{\mathcal{B}_{x_n}^0 \mathcal{B}_{x_n}^0 \ldots \mathcal{B}_{x_n}^0} \right| \Delta \lambda, \\
\end{align*}
\]

(5.6)

\[
- 16 -
\]
and
\[
\text{the incremental output current } I^1 = \left\{ \prod_{i=1}^{l} \left[ \frac{S_{x_i}}{S_{x_i}^c} \right] \right\} \Delta s.
\]

Therefore the maximum amount of incremental input current \( \Delta s \) along such a route is given by the smaller of
\[
\min_{i=1} \left\{ \frac{B_{x_i}^c \cdots B_{x_{i+1}}^c}{B_{x_i} \cdots B_{x_{i+1}}} \left( c^{x_i} \right) \right\}
\]
and
\[
\min_{i=1} \left\{ \frac{B_{x_i}^c \cdots B_{x_{i+1}}^c}{B_{x_i} \cdots B_{x_{i+1}}} \left( x^{x_i} \right) \right\}
\]  \( \text{(5.7)} \)

The conduction route of the type shown in Fig. 8(b) is a route for which
\[
\begin{align*}
T_{x_i} & = T_i \quad \text{(input),} \\
q_{x_i}^0 & = q_i^0 \quad \text{(a fixed node, } l \neq i \neq j), \\
T_{x_i} & \neq T_{x_j} \\
& \quad \text{if } i \neq i, \neq j, \neq j,
\end{align*}
\]  \( \text{(5.8)} \)
every branch contained with positive (negative) sign is conductive in the positive (negative) direction, and \( \text{(5.9)} \) holds.
\[
\beta = \left| \frac{B_{x_i}^{u_{i+1}} \cdots B_{x_{i+1}}^{u_{i+2}} \cdots B_{x_{i+2}}^{x_i} \cdots B_{x_{i+2}}^{x_i}}{B_{x_i}^{x_i} \cdots B_{x_{i+1}}^{x_i} \cdots B_{x_{i+1}}^{x_i}} \right| < 1.
\]  \( \text{(5.9)} \)

A sequence of nodes and (directed) branches for which \( \text{(5.10)} \) holds:
\[
\begin{align*}
T_{x_i} & = T_i \quad \text{(input),} \\
q_{x_i}^0 & = q_i^0 \quad \text{(a fixed node, } l \neq i \neq j), \\
T_{x_i} & \neq T_{x_j} \\
& \quad \text{if } i \neq 0, \neq k, \neq j,
\end{align*}
\]

instead of \( \text{(5.3)} \), is called a "loop". Thus the conduction route of the type of Fig. 8(b) contains, as its final part, a loop. We shall call the \( \beta \) defined along a conduction

1) Note that this is not in general equal to the incremental input current \( \Delta s \).
loop (i.e., a loop whose branches are conductive in the proper directions, respectively) by the same formula as (5.9) the "feedback constant" along the loop. Moreover, we shall call a loop along which the feedback constant is less than 1 (such as the part of the conduction route of the type Fig. 8(b)) a "suction loop" (the reason for this nomenclature will be obvious). The branch currents along the route of the type Fig. 8(b) are determined successively (if the current entering the input node \( J_i \) is \( \Delta S > 0 \)):

\[
\varepsilon_i \Delta x_i = \begin{vmatrix}
\delta^0, \delta^1, \ldots, \delta^{i-1} \\
\delta_i, \delta^1, \ldots, \delta^{i-1}
\end{vmatrix} \Delta l \\
\varepsilon_i \Delta x_i = \begin{vmatrix}
\delta^0, \delta^1, \ldots, \delta^{i-1} \\
\delta_i, \delta^1, \ldots, \delta^{i-1}
\end{vmatrix} \frac{\Delta S}{1 - \beta}
\]

\( (1 \leq i \leq i_0), \quad (i_0 < i \leq 0) \). \quad (5.11)

Therefore the maximum amount of incremental input current \( \Delta S \) along such a conduction route is given by the smallest of

\[
\min_{\varepsilon_i, \varepsilon_i = 1} \begin{vmatrix}
\delta^0, \delta^1, \ldots, \delta^{i-1}, \delta^i \\
\delta^0, \delta^1, \ldots, \delta^{i-1}
\end{vmatrix} (c^x_i - \delta x_i),
\]

\[
\min_{\varepsilon_i, \varepsilon_i = 1} \begin{vmatrix}
\delta^0, \delta^1, \ldots, \delta^{i-1}, \delta^i \\
\delta^0, \delta^1, \ldots, \delta^{i-1}
\end{vmatrix} \delta x_i,
\]

\[
\min_{\varepsilon_i, \varepsilon_i = 1} \begin{vmatrix}
\delta^0, \delta^1, \ldots, \delta^{i-1}, \delta^i \\
\delta^0, \delta^1, \ldots, \delta^{i-1}
\end{vmatrix} (1 - \beta)(c^x_i - \delta x_i),
\]

\( (5.12) \)

\[1) \text{Important is the fact that the feedback constant is invariant under the transformation} \{ \tau^0_0, \tau^1_0, \ldots, \tau^0_1, \tau^0_0, \tau^1_0, \ldots, \tau^0_1, \tau^0_0 (= \tau^0_0) \} \rightarrow \{ \tau^0_0, \tau^0_1, \ldots, \tau^0_1, \tau^0_0 (= \tau^0_0), \tau^0_0, \ldots, \tau^0_0, \tau^0_0 \}. \]
and

\[
\min_{\beta} \left| \begin{array}{cccc}
S_k, & S_k^2, & \cdots & S_k^{n-1}, \\
S_k^2, & S_k^3, & \cdots & S_k^{n-2}, \\
\vdots & \vdots & \ddots & \vdots \\
S_k^{n-1}, & S_k^{n-2}, & \cdots & S_k^2
\end{array} \right| (1-\beta) \lambda_k^2
\]

If a conduction route (containing a suction loop or not) is found, we assign the maximum possible amount of current determined by (5.6) and (5.7) or (5.11) and (5.12) along the route. Obviously the current configuration obtained by superposing the incremental currents to the given one satisfies (2.1) - (2.4) with respect to the given voltage configuration. Through this process, the states of branches are altered as follows:

\[
A \rightarrow A, \quad B \rightarrow B, C \text{ or } D, \quad C \rightarrow B, C \text{ or } D, \\
D \rightarrow B, C \text{ or } D, \quad E \rightarrow E.
\]

(5.13)

The change (increase) in the cost \( z = e_k S_k \) is given by

\[
\Delta z = z(S + \Delta S) - z(S) = u_1 \Delta S
\]

(5.14)

in view of (3.1) with the inequality replaced by an equality. If assignment of incremental current has been made along a conduction route, we regard the new current configuration obtained by superposition as the given configuration and repeat the same process as the above. Then we seldom fail to reach the situation, after a finite number of repetitions, where we can no longer find any conduction route, or where the infinite amount of incremental current can be assigned along a conduction route.¹)

(See §8. ) If the latter situation is reached, the whole

1) To secure the finiteness of the number of repetition, we must resort to a little more complicated rule; i.e. we must regard the branches, which have been brought in state B or D from another state, as non-conductive so long as we can find conduction routes under this additional restriction. If we encounter the situation where no conduction route can be found under this additional restriction, we obliterate the past history of the branches, restarting anew. (In this respect, cf. [7].)
solution process ends. It the former situation is reached, we proceed to a voltage-increasing step with the obtained current configuration and the given voltage configuration as the given set of current and voltage configurations for the next voltage-increasing step.

Through a current-increasing step the change in the states of branches and that in the total cost are obviously given by the same formulae as (5.13) and (5.14).

Note: We may search for a conduction route according to any method. The problem is regarded as a slight generalization of the labyrinth problem. While it seems possible to generalize the labelling method of Ford and Fulkerson[8], [9],[10] so as to fit for this problem, it will be an easy task to search for a conduction route, especially after a voltage-increasing step has finished, by systematically tracing conductive branches and computing each time by (5.6) with $\Delta S = 1$. The following method, shown in Fig.9 in the form of a flow chart, will be one such example.

In Fig.9, if we start from START, after a finite number of steps we either arrive at STOPA or at STOPB.

<Explanation of Fig.9 on the following page>

d(a,b): discrimination whether there exists a branch $\gamma_k$ such that $(\tau_{a}^k - \tau_{b}^k)$ and that it is conductive from $\tau_{a}^k$ to $\tau_{b}^k$, or not.

(It can be assumed without loss of generality that, given a pair of nodes $\tau_{a}^k$ and $\tau_{b}^k$, there exists at most one branch $\gamma_k$ which corresponds to $d(a,b) = YES$, so that we have $\xi = \xi(a,b)$ (=1 or -1) and $\chi = \chi(a,b)$ (=1, 2, ..., or n) as functions defined over the set of node pairs such that $d(a,b) = YES$. Furthermore, in accordance with the footnote on p.8, we assume $d(a,a) = NO$ for all a.)

c1, ..., cm: $m (= \text{the number of nodes})$ main cycle counters which, being the storages for the node numbers along the conduction route, may count from 0 to $m+1$

$\beta_1, ..., \beta_m$: $m$ storages for current amplification (i.e. for $\xi \Delta S_i / \Delta S$)

p, q, r: three auxiliary storages.
In the former case, we obtain a conduction route

(i) $J_i^0 = J_{c_i}^0$, $\varepsilon_i J_{c_i}^0$, $J_{c_i}^0$, ..., $J_p^0$, $\varepsilon_p J_{c_p}^0$, $J_{c_p}^0 = J_m^0$

($\varepsilon_i = \varepsilon(c_i, c_{i+1})$, $\chi_i = \chi(c_i, c_{i+1})$,
$p = \text{length of the route, } q = 0$)

or

(ii) $J_i^0 = J_{c_i}^0$, $\varepsilon_i J_{c_i}^0$, $J_{c_i}^0$, ..., $J_e^0$, $\varepsilon_e J_{c_e}^0$, ..., $\varepsilon_p J_{c_p}^0$, $J_{c_p}^0 = J_m^0$

along which the maximum incremental current can easily be determined. In the latter, there exists no conduction route.

In Theorem 5 of §7, we shall prove that STOP is never reached in the current-increasing step which follows the completion of a voltage-increasing step.
5.2. Voltage-increasing steps. Let us suppose that a pair of a current and a voltage configuration is given, together with \( B \) and \( z = e_x S_x \), which satisfies (2.1) - (2.4). Then, we require to increase the value of \( u_1 \) as high as possible with all the currents fixed. Our basic idea about the solution of this problem is the same as in [4]. Instead of algebra \( \mathcal{A} \), we now make use of algebra \( \mathcal{A}' \), whose definition as well as fundamental properties is given in §4 (Chapter I) of [3]. As the characteristic of a branch in state \( A, B, C, D \) or \( E \), we define

\[
\begin{align*}
\beta_x & \overset{\text{def}}{=} \beta(\varepsilon^x, |\delta_x^a|, |\delta_x^b|) \quad (\varepsilon \in \mathcal{A}'), \\
\beta_x' & \overset{\text{def}}{=} (\varepsilon^x, |\delta_x^a|, |\delta_x^b|) \quad (\varepsilon \in \mathcal{A}'),
\end{align*}
\]  

(5.15)

where

\[
\varepsilon^i_x = \varepsilon^i_x^+ - \varepsilon^i_x^-,
\]

\[
\varepsilon^i_x = \begin{cases} 
\infty & \text{when } \varepsilon^i_x \text{ is in state } D \text{ or } E, \\
-e & \text{when } \varepsilon^i_x \text{ is in state } A, B \text{ or } C,
\end{cases}
\]

(5.16)

\[
\varepsilon^i_x = \begin{cases} 
\infty & \text{when } \varepsilon^i_x \text{ is in state } A \text{ or } B, \\
e & \text{when } \varepsilon^i_x \text{ is in state } C, D \text{ or } E,
\end{cases}
\]

and \( \beta(c,d,e) \) is an operator, which, operating on \( x \in \mathcal{A}' \) (real number or \( \infty \)), yields

\[
\beta(c,d,e).x = \frac{ex+c}{d}.
\]

(5.17)

Then we define the one-step transmission matrix \( B \) (representing the node-to-node characteristics) by

\[
B = (\varepsilon^a_x): \quad \varepsilon^a_x \overset{\text{def}}{=} \begin{cases} 
\beta(0,1,1) & \text{for } a=b, \\
\varepsilon^a_x((-D_x^b)\cdot|D_x^b|) & \text{for } a \neq b,
\end{cases}
\]

(5.18)

where

\[
D_x^a = \text{sgn } \delta_x^a, \quad D_x^b = \text{sgn } \delta_x^b,
\]

(5.19)
\[ \beta_x(0) \overset{\text{def}}{=} \beta(-1, 0, 1), \]
\[ \beta_x(1) \overset{\text{def}}{=} \beta_x, \]
\[ \beta_x(-1) \overset{\text{def}}{=} \beta_x^{\text{ext}}, \]
\[ (\beta \cdot \beta')_x \overset{\text{def}}{=} \min(\beta \cdot x, \beta' \cdot x) \]
\[ (x: \text{ real number or } \infty). \]

The multiplication of \( B \) by a vector whose components are real numbers or \( \infty \) is defined by
\[ B \cdot \psi \overset{\text{def}}{=} \left[ \min_a \beta_i^a \psi_a \right], \]
\[ \psi = \left[ \psi_b \right], \]
from which follows the relation:
\[ B' \cdot (B \cdot \psi) = (B' \cdot B) \cdot \psi, \]
where the law of composition \( \cdot \) is the same as defined in §4.4 of [4]. (It should be noted that all the above procedures are in exact conformity with those adopted for general information networks in [4]. Their physical meanings will be so obvious that no further explanation may be needed.)

The new voltage configuration which we require is obtained by the following iterative process. First, we put
\[ \psi_0 = (\hat{v}_b), \quad \psi_b = \left\{ \begin{array}{ll} 0 & \text{for } \mathcal{V}_b = \mathcal{V}_m, \\ M & \text{for } \mathcal{V}_b \neq \mathcal{V}_m, \end{array} \right. \]
where \( M \) is a sufficiently large, but finite, positive real number. Then we repeat the calculation according to (5.25)
\[ v_{k+1} = (B \cdot B \cdot \cdots \cdot B) \cdot \psi = B \cdot \psi_k \quad (k = 0, 1, 2, \cdots), \]

1) See the footnote on p. 28.
putting
\[ u \overset{\text{def}}{=} \lim_{k \to \infty} \nu_k \] (5.26)
In order to show the legitimacy of the definition (5.26), we first note that
\[ k \nu = \min_a (\beta_b a) = \beta_b a \nu_a \leq \beta_b (0,1,1) \nu = \frac{1 \cdot \nu_b}{1} = k \nu_b, \]
or
\[ k \mu \leq \mu \leq \nu \leq \nu_b \leq \nu_b \quad \text{for all } b. \] (5.27)
The sequence \( \{\nu\} \) is, therefore, non-increasing. Furthermore we can show (see Lemma given just below) that it has a lower bound \( \mu \) (the given node voltages). Hence it has the limit \( \mu \). The \( \mu \) thus determined satisfies the inequality relation:
\[ (M) \geq \mu \geq \mu \geq (0), \] (5.28)
where \( \mu \) is the vector consisting of the given node voltages, \( M \) or \( 0 \) is the vector whose components are all 1 or 0, and the last inequality follows by induction from the fact (see §6) that the first voltage configuration we started the solution process from was such that \( \mu = 0 \).

**Lemma.** All the \( \nu \)'s (\( k=0,1,2, \cdots \)), hence also their limit \( \nu \), obtained in the course of a voltage-increasing step are not smaller than the given \( \mu \) :
\[ \nu \geq \mu \quad (k=0,1,2, \cdots), \]
\[ \mu \geq \mu, \] (5.29)
if \( M \) is sufficiently large.

**Proof:** Let \( \{ \nu_k = \nu_k^0, \nu_k^1, \cdots, \nu_k^\ell_k, \nu_k^0 = \nu_k^0 \} \) be a sequence of nodes such that
\[ \nu_k = \min (\beta_{c_k} \nu_c) = \beta_{c_k} \nu_c \quad (\ell=1,2, \cdots, k), \] (5.30)
1) The case where \( c_{c_k}^0 = c_{c_k}^0 \) for an \( l \) may not be considered, for, if such is the case, we can find a \( k' < k \) such that \( c_{c_k}^0 = c_{c_k}^0 \) and make the following discussion in regard to \( \nu_k \).
and in addition, let \( \{ \varepsilon, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k \} \) be a sequence of branches such that
\[
\varepsilon_k \leq \mathcal{G}^k - \mathcal{C}^k = \mathcal{C}^k_{i-1} \quad (\varepsilon_i = 1 \text{ or } -1)
\]
and that
\[
\nu_{C_i} = \mathcal{B}_{C_i} (-\varepsilon_i) \cdot \nu_{C_{i-1}}.
\]
Since \( \nu \) satisfies (2.3),
\[
\theta_{\varepsilon_i} = \begin{cases} 
-e_i & \text{if and only if } E_{\varepsilon_i} \geq 0 \text{ and } E_{\varepsilon_i} = 0, \\
\infty & \text{otherwise,}
\end{cases}
\]
\[
\theta_{\varepsilon_i} = \begin{cases} 
e_i & \text{if and only if } E_{\varepsilon_i} \geq 0 \text{ and } E_{\varepsilon_i} = 0, \\
\infty & \text{otherwise,}
\end{cases}
\]
where \( E_{\varepsilon_i} \) and \( E_{\varepsilon_i} \) are determined from \( \nu \) (see (2.3) and (5.16)). Hence, for an arbitrary branch \( \mathcal{G}_{\varepsilon_i} \),
\[
\mathcal{B}_{\varepsilon_i} \mathcal{G}^{\varepsilon_i} \mathcal{S}_{\varepsilon_i} = e_{\varepsilon_i} + (E_{\varepsilon_i} - E_{\varepsilon_i^0}) \leq \mathcal{G}^{\varepsilon_i}
\]
i.e., if \( \exists \mathcal{G}_{\varepsilon_i} = \mathcal{G}_{\varepsilon_i}^0 - \mathcal{G}_{\varepsilon_i}^0 \),
\[
\mathcal{B}_{\varepsilon_i} (-\mathcal{B}_{\varepsilon_i}^{\varepsilon_i}) \mathcal{C}_{\varepsilon_i} = \mathcal{B}_{\varepsilon_i} (-\varepsilon_i) \cdot \mathcal{C}_{\varepsilon_i} \leq \mathcal{C}_{\varepsilon_i}
\]
\[
\beta_{\varepsilon_i} \mathcal{B}_{\varepsilon_i} (-\mathcal{B}_{\varepsilon_i}^{\varepsilon_i}) \mathcal{C}_{\varepsilon_i} = \mathcal{B}_{\varepsilon_i} (-\varepsilon_i) \cdot \mathcal{C}_{\varepsilon_i} \leq \mathcal{C}_{\varepsilon_i}
\]
Therefore, if
\[
\nu_{C_{i-1}} \geq \mathcal{G}_{C_{i-1}}
\]
then
\[
\nu_{C_i} = \mathcal{B}_{\varepsilon_i} (-\varepsilon_i) \cdot \nu_{C_{i-1}} \geq \mathcal{B}_{\varepsilon_i} (-\varepsilon_i) \cdot \mathcal{C}_{C_{i-1}} \geq \mathcal{C}_{C_i}.
\]
On the other hand, if \( M \) is sufficiently large,
\[ v_a \geq u_a \quad \text{for all } a, \quad (5.38) \]

from which (5.29) follows by induction.

Let us prove in the following two theorems that the new node voltages \( u_b \)'s (components of \( u \)'), as well as \( E_x \)'s and \( E'_x \)'s defined by

\[ E_x - E'_x = B_x^a u_a - e_x, \quad E_x E'_x = 0, \quad (5.39) \]

are the required ones.

**Theorem 1.** The new voltage configuration determined by (5.26) and (5.39) satisfies (2.3) and (2.4) in regard to the given current configuration.

**Proof:** From (5.25) and (5.26) it follows that

\[ B \cdot u = u \quad \text{or} \quad m_a^b (\beta^b \cdot u) = u_b \quad \text{for any } b. \quad (5.40) \]

Hence we have, for any branch \( \sigma^x \) (we assume that \( \sigma^{x'} = \sigma^x - \sigma^b \)),

\[ |D_{x'}^{u} u_{(a)} + \theta^x| \geq |D_{x'}^{u} u_{(b)} + \theta^x| \geq u_a, \quad (5.41) \]

or

\[ |D_{x'}^{u} u_{(a)} - |D_{x'}^{u} u_{(b)} + \theta^x| = D_{x'}^{u} u_c + \theta^x \geq 0, \quad (5.42) \]

Substitution of (5.39) in (5.42) gives

\[ E_x - E'_x + \left( \theta^x + e_x \right) \geq 0, \quad (5.43) \]

namely,

\[ \theta^x - e_x \geq E_x - E'_x = \left( \theta^x + e_x \right). \quad (5.44) \]

---

1) Note that always \( \dot{v}_m = u_m = 0. \)
Comparing (5.44) with (5.16), we conclude that

\[ E'_x - E'_x \geq 0, \text{ hence } E'_x = 0, \]

if \( \mathcal{T}_k \) was in state \( A, B \) or \( C \) in regard to the given current and voltage configuration, i.e., if \( S^x > 0 \),

and

\[ E'_x - E'_x \leq 0, \text{ hence } E'_x = 0, \]

if \( \mathcal{T}_k \) was in state \( C, D \) or \( E \) in regard to the given current and voltage configuration, i.e., if \( S^x < c^x \),

which condition is contrapositive (hence, equivalent) to (2.4). The validity of (2.3) is obvious.

**Theorem 2.** The new \( u_b \) at node \( \mathcal{T}_k \), hence in particular \( u_1 \), is not less than the \( u_b \) of any other voltage configuration satisfying (2.3) and (2.4) in regard to the given current configuration, unless it is possible for \( u_b \) to become infinitely large.

**Proof:** First, it is noticed that the values of \( u_b \)'s depend on the given current configuration, but not on the given voltage configuration. Hence it suffices to prove that a new \( u_b \) is not less than the given \( u_b \). This, however, is evident from the above lemma and the definition of \( u_b \).

If \( u_1 = o(M) \) is obtained for sufficiently large \( M \), we proceed to a current-increasing step.

If \( u_1 = O(M) \) is obtained\(^1\), the solution process terminates, because \( u_1 \) can then be made infinitely large.

---

\(^1\) As will gradually become clear in the following section, the value of a \( u_b \) (unless it can be made infinitely large) has an upper bound. We may choose as \( M \) a number greater than all such upper bounds, determining whether \( = o(M) \) or \( = O(M) \) according to the relative magnitude to such upper bounds. But it will be more practical and theoretically rigorous to introduce the symbol \( M \) standing for such a number, assuming the properties of \( M \)

(continued on the next page)
and $S$ cannot be increased any more (cf. Theorem 9 in §7). We put $\mathbf{u_1} = \infty$ in this case.

Obviously no change occurs in $z = e^x S^x$ through a voltage-increasing step, from which the invariance of $w = S\mathbf{u_1} - c^x E_x$ also follows at once:

$$w = S'\mathbf{u_1} - c^x E_x = z(S) = S\mathbf{u_1} - c^x E_x = w,$$

(5.45)

since both the old voltage configuration $\mathbf{u}$ and the new $\mathbf{u}'$ satisfy (2.3) and (2.4) in regard to the common, namely given, current configuration.

(continued from the previous page)

such as explained in §9.2. The precise meaning of $o(M)$ or $O(M)$ then becomes as follows: $\mathcal{C} = o(M)$ if $\mathcal{C} = x'$ and $\mathcal{C} = O(M)$ if $\mathcal{C} = XM + x'$ where $\infty > x > 0$ and $\infty > x' \geq 0$. We encounter a similar situation when introducing the penalty $M$ in ordinary linear programming theory [11], [12].
6. Starting the Solution Process

We may start the solution algorithm with any pair of a current configuration and a voltage configuration satisfying (2.1) - (2.4), but it is advisable to start from the configuration:

\[ S = S^* = u_b = E_x = 0 \quad \text{and} \quad E'_x = e_x \quad (6.1) \]

for all \( x \) and all \( b \). This configuration obviously satisfies (2.1) - (2.4).
7. Details concerning the Solution Process

In this section we shall show, in the form of Theorems, that the two kinds of steps — current-increasing and voltage-increasing — explained in §5 are sufficient in order to solve the problem defined in §2.

Lemma.

Theorem 1.

Theorem 2.

Theorem 3. Let us denote by \(<C_\ell>\) and \(<V_\ell>\) the \(\ell\)-th current-increasing and the \(\ell\)-th voltage-increasing step, assuming that \(<V_\ell>\) follows \(<C_{\ell-1}>\) and \(<C_\ell>\) follows \(<V_{\ell-1}>\), and, furthermore, let us denote by \(s\) and \(u_1\) the \(s\) and the \(u_1\) obtained through \(<C_{\ell}>\) and \(<V_{\ell}>\) respectively. Then

\[
0 \leq u_1 < u_2 < \ldots < u_k < \ldots < u_N \leq \infty,
\]

\[
0 < s_1 < s_2 < \ldots < s_k < \ldots < s_N = \infty;
\]

\( (7.1.1) \)

or

\[
0 \leq u_1 < u_2 < \ldots < u_k < \ldots < u_N \leq u = \infty,
\]

\[
0 < s_1 < s_2 < \ldots < s_k < \ldots < s_N < \infty;
\]

\( (7.1.2) \)

and

\[
\frac{d}{ds} z(s) = u_1 \quad \text{if} \quad s_1 < s < s_1, \quad (7.2)
\]

where \(N\) is the number of current-increasing steps repeated during the whole solution process.

Proof: \((7.2)\) is a mere reformulation of \((5.14)\) (See also Theorem 4). If all the equalities and the inequalities are replaced by \(\leq\), \((7.1.1)\), and \((7.1.2)\) are obvious from the definition of \(<C>\)'s and \(<V>\)'s. Further refinement (i.e., < or = instead of \(\leq\)) will be made in the following theorems and the proof of the existence of a finite \(N\) will be made in the following section.

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Theorem 4. Let \( S_1, S^*; y_b, E_x, E'_x \) and \( S_2, S^*_2; y_b, E_x, E'_x \) be two pairs of current and voltage configurations, each satisfying (2.1) - (2.4). If \( S = S_2 \)
and \( S^* = S^*_2 \), then the voltage configuration
defined by
\[
\begin{align*}
\nu_a &= \lambda \nu_a + \mu \frac{\nu_a}{2} \\
E_x &= \lambda E_x + \mu E_x \\
E'_x &= \lambda E'_x + \mu E'_x
\end{align*}
\]
also satisfies (2.1) - (2.4) in regard to the current configuration determined by \( (S = S_2; S^* = S^*_2) \).
Likewise, if \( y_b = y_b; E_x = E_x \) and \( E'_x = E'_x \), then the current configuration defined by
\[
\begin{align*}
S &= \lambda S + \mu S \\
S^* &= \lambda S^* + \mu S^*
\end{align*}
\]
also satisfies (2.1) - (2.4) in regard to the voltage configuration determined by \( (y_b = y_b; E_x = E_x; E'_x = E'_x) \).

Proof: The theorem is obvious because all the equations and inequalities in (2.1) - (2.4) are linear in regard to the set of variables \( (S, S^*) \) as well as to the set \( (y_b, E_x, E'_x) \).

Theorem 5. If a voltage-increasing step has finished with a finite \( u_1 \), in the next current-increasing step \( S \) can certainly be increased.

Proof\(^{1)}\): Here "finite" means "o(M)".\(^2)\) In order to prove the theorem it suffices to show that \( \text{STOP}_B \)
is never reached when we search for a conduction route according to Fig.9 (see p.21) after a voltage-increasing step. Let us assume to the contrary that \( \text{STOP}_B \) was reached, and denote the set of nodes which had appeared as \( T_r \) by \( N \) and the

\(^{1)}\) In this proof here, Einstein's summation convention is not used.

\(^{2)}\) See footnote \(^{1)}\) on p.28 - 29.
rest by \( \mathcal{Y} \). 1) Since \( \mathcal{C}_i = \mathcal{C}_i \) (see Fig.9) and \( \mathcal{E}_i \) never appears as \( \mathcal{C}_i \) by assumption, neither \( \mathcal{Y}_i \) nor \( \mathcal{R}_i \) is void, i.e., \( \mathcal{C}_i \in \mathcal{Y}_i \) and \( \mathcal{C}_i \in \mathcal{R}_i \). Our assumption of the arrival at STOP further requires the \( \beta \)'s (feedback constant) around the conduction loops all of whose nodes belong to \( \mathcal{R}_i \), not to be less than 1. Since the network is assumed to be connected there exists the cut-set defined by

\[
\delta \left( \sum \mathcal{C}_i \right) = -\delta \left( \sum \mathcal{C}_i \right) = \sum_{\mathcal{C}_i \in \mathcal{Y}_i} \varepsilon_{\mathcal{C}_i}, \tag{7.5}
\]

where \( \delta \) is the set of branches belonging to the cut-set and \( \varepsilon_{\mathcal{C}_i} \) is 1 or -1 according as the branch \( \mathcal{C}_i \) belongs to the cut-set with positive sign or with negative sign. Let us note here that there exists a conductive branch from \( \mathcal{C}_i \) to \( \mathcal{C}_j \) \((\neq \mathcal{C}_i)\) if and only if \( \mathcal{V}_b = \beta_{ij} \mathcal{V}_a \), and then investigate the manners in which the sequence \( \{ \mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_1, \cdots \} \) converges to \( \mathcal{V}_1 \).

(i) The first case:

\[
\mathcal{V}_1 = \lim_{k \to \infty} \mathcal{V}_1 = \mathcal{V}_1 \quad \text{for some } k_0. \tag{7.6}
\]

In this case we take as \( k_0 \) the least integer satisfying \( \mathcal{V}_1 = o(M) \) and \( \mathcal{V}_1 = M \), \( k_0 \) must be greater than or equal to 1. Furthermore, we shall have

\[
\mathcal{V}_1 = \mathcal{V}_1 = (\beta_{1}^{\mathcal{C}_i} \cdots \beta_{1}^{\mathcal{C}_i}) \mathcal{V}_1 = (\mathcal{V}_1 = 0, \mathcal{C}_i \neq \mathcal{C}_i) \tag{7.7}
\]

for a certain sequence of nodes \( \{ \mathcal{C}_i = \mathcal{C}_i, \mathcal{C}_i, \cdots, \mathcal{C}_i \} \), for, if otherwise

\[
\mathcal{V}_1 = \mathcal{V}_1 = (\beta_{1}^{\mathcal{C}_i} \cdots \beta_{1}^{\mathcal{C}_i}) \mathcal{V}_a (a \neq M, \mathcal{V}_a = M), \tag{7.8}
\]

we have

\[
\mathcal{V}_1 = \lambda M + \mathcal{V}_1 \quad \text{with } \lambda > 0, \tag{7.9}
\]

which contradicts the assumption that \( \mathcal{V}_1 = o(M) \). From (7.7) it readily follows that

---

1) In brief, \( \mathcal{Y}_i \) is the set of those nodes to which there can be found a conduction route from \( \mathcal{C}_i \), and \( \mathcal{R}_i \) is the set of those nodes to which no such route can be found.
'u_i = \nu_i = (\beta_i^{c_i} \perp \cdots \perp \beta_j^{c_j}) \cdot u_m, \quad (7.10)

since we have the following cycle of inequalities:

\[ u_1 \leq (\beta_i^{c_i} \perp \cdots \perp \beta_j^{c_j})'u_i \leq (\beta_i^{c_i} \perp \cdots \perp \beta_j^{c_j})'v_m \]

\[ 'v_m \leq (\beta_i^{c_i} \perp \cdots \perp \beta_j^{c_j})'u_1 = 'u_1. \quad (7.11) \]

However, the second equation of (7.10) implies the existence of conduction from \( \mathcal{J}_1^o \) to \( \mathcal{J}_1^o \), which further implies the existence of a conduction route from \( \mathcal{J}_1^o (\in \mathcal{X}_1) \) to \( \mathcal{J}_m^o (\in \mathcal{X}_2) \), thus giving rise to a contradiction.

(ii) The second case: 

'\( u_1 < \frac{k}{v_1} \) for any finite \( k \). \quad (7.12)'

In this case,

'\( u_b < \frac{k}{v_b} \) for any finite \( k \)

and any \( \mathcal{J}_b^o \in \mathcal{Y}_1 ; \quad (7.13) \)

for, otherwise, we shall have

\[ 'u_b = \frac{k}{v_b} \text{ for a finite } k \text{ and a } \mathcal{J}_b^o \in \mathcal{Y}_s, \quad (7.14) \]

and, along a conduction route \{ \( \mathcal{J}_1^o = \mathcal{J}_s^o, \mathcal{J}_c^o, \cdots, \mathcal{J}_s^{c_i}, \mathcal{J}_c^s = \mathcal{J}_b^s \} \)

\[ \frac{k^{+j}}{v_1} \leq (\beta_i^{c_i} \perp \cdots \perp \beta_j^{c_j})'v_b \]

\[ = (\beta_i^{c_i} \perp \cdots \perp \beta_j^{c_j})'u_b = 'u_1. \quad (7.15) \]

and hence

\[ 'u_1 = \frac{k^{+j}}{v_1}, \quad (7.16) \]

1) Note that \[ 'u_i = \beta_i^{c_i} 'u_i \quad (i=1,2,\cdots,l) \] along the conduction route.
which last equation contradicts (7.12). Moreover, since there exists no conduction from a node \( \gamma_b \in \gamma' \) to a node \( \gamma_a \in \gamma' \),

$$u_b \neq \beta_b u_a \text{ for any } \gamma_b \in \gamma', \text{ and any } \gamma_a \in \gamma' \cdot$$

(7.17)

Therefore, because \( \lim_{k \to \infty} v_b = u_b \), there exists a number \( N \) such that, if \( k \geq N \),

$$v_{b'} \neq \beta_{b'} v_a \text{ for any } \gamma_{b'} \in \gamma', \text{ and any } \gamma_a \in \gamma' \cdot$$

(7.18)

Next, let us put

$$v_b - u_b = w_b \cdot$$

(7.19)

Then we have

$$\lim_{k \to \infty} w_b = 0 \text{ for all } b \cdot$$

(7.20)

and

$$w_b \geq k+1 \cdot$$

(7.21)

It also follows from the very definition of \( \epsilon^a_b \)'s that

if \( v_b = \beta^a_b v_a \) then

$$w_b \geq |\beta^a_b| w_a \cdot$$

(7.22)

where \( \gamma_{b'} \) is the branch such that

$$v_{b'} = \pm (\gamma_b - \gamma_{b'}) \text{ and } \beta^a_b = \beta_+ \text{ or } \beta_- \cdot$$

(7.23)

Finally, let us consider the subsequence \( \{ \hat{v}_1, \hat{v}_1, \hat{v}_1, \ldots \} \) of the sequence \( \{ \check{v}_1, \check{v}_1, \check{v}_1, \ldots \} \), composed of all \( \check{v}_1 \)'s such that

$$v_1 \geq v_1 \geq \check{v}_1 \geq v_1 \geq \check{v}_1 \geq \ldots \cdot$$

1) In fact, always \( u_b \leq \beta_b u_a \).
This subsequence must be infinite owing to (7.13). Then, for any \( \varepsilon > 0 \), there exists a positive integer \( N' \) such that, if \( k \geq N' \) and \( N + k \) coincides with one of the \( kj \)'s, a sequence of nodes \( \{\nu_{j}^{\varepsilon}, \nu_{k}^{\varepsilon}, \ldots, \nu_{k-1}^{\varepsilon}, \nu_{k}^{\varepsilon}\} \) can be found for which

\[
\begin{align*}
N + k \quad & \nu_{1} = u_{1}^{\varepsilon}, \\
N + k \quad & v_{1}^{\varepsilon} = (\beta \varepsilon_{1}^{\varepsilon} \cdots \beta \varepsilon_{k-1}^{\varepsilon})v_{c_{k}}, \\
N + k \quad & v_{c_{i} - 1}^{\varepsilon} = \varepsilon_{c_{i} - 1}^{\varepsilon} v_{c_{i}} (i = 1, 2, \ldots, k - 1, k), \\
\varepsilon_{c_{i}^{\varepsilon}} & \in \mathcal{T}_{c_{i}}, \quad \varepsilon_{c_{i}^{\varepsilon}} \neq \varepsilon_{c_{i}}.
\end{align*}
\]

(7.25)

Along this sequence with a properly chosen set of branches \( \{\varnothing_{c_{i}}, \ldots, \varnothing_{c_{k}}\} \), we have

\[
N + k + 1 \quad w_{c_{i} - 1}^{\varepsilon} = \begin{vmatrix} \frac{\delta_{c_{i}^{\varepsilon}}}{\varepsilon_{c_{i}^{\varepsilon}}} & N + k + 1 \quad w_{c_{i}}^{\varepsilon} \end{vmatrix}
\]

(7.26)

by virtue of (7.22), and consequently,

\[
N + k \quad w_{1}^{\varepsilon} = \begin{vmatrix} \delta_{c_{1}^{\varepsilon}} & \ldots & \delta_{c_{k}^{\varepsilon}} \\
\varepsilon_{c_{1}} & \ldots & \varepsilon_{c_{k}} \end{vmatrix} w_{c_{k}}^{\varepsilon}
\]

def \( \varphi (c_{i}, \varepsilon_{i}^{\varepsilon}) \) \( N w_{c_{k}}^{\varepsilon} \).

(7.27)

If we put

\[
\min_{c_{i} \in \mathcal{T}_{c_{i}}} \eta_{c_{i}} (\gamma > 0),
\]

(7.28)

we have

\[
\varepsilon > N + k \quad w_{1}^{\varepsilon} \geq \varphi (c_{i}, \varepsilon_{i}^{\varepsilon}) \gamma > 0.
\]

(7.29)

Since \( \varepsilon \) can be chosen arbitrarily small while \( \gamma \) is fixed, we must have

\[
\lim_{k \to \infty} \max_{\gamma} \varphi (c_{i}, \varepsilon_{i}^{\varepsilon}) = 0,
\]

(7.30)

where \( \max \) is taken over all the sequences of nodes of length \( k ( = kj - N ) \), which satisfy (7.25), with
the corresponding sequences of branches \( \{ \xi_k^l, \cdots, \xi_k^l \} \) such that
\[
\gamma \xi_k^l = \pm (\xi_k^l - \xi_k^l) \quad \text{and} \quad \rho_k^l = \rho_k^l \text{ or } \rho_k^l.
\] (7.31)

If we denote the feedback constant around an irreducible (i.e., containing no proper subloop) loop, which is not necessarily a conduction loop with regard to \( \tau \), by \( \bar{\beta} \) \( (\nu = 1, 2, \cdots, 1) \) and the current amplification factor along an irreducible (i.e., containing no loop) path, which is not necessarily a conduction route with regard to \( \tau \), by \( \varphi \) \( (\mu = 1, 2, \cdots, 1) \), where all the nodes contained in the loops and paths under consideration are assumed to belong to \( \tau \), then we shall have such an expression of \( \varphi \) as follows:
\[
\max \varphi = \prod_{\mu}^{\nu} \varphi^{\nu}_{\mu} \prod_{k}^{\nu} \beta^{\nu}_{\mu},
\] (7.32)
\[(k \text{ fixed}) \]

where
\[
\omega_{\nu} = 0 \text{ or } 1,
\]
\[
\lambda_{\nu} = 0 \text{ or a positive integer.}
\] (7.33)

Obviously, for at least one \( \nu \),
\[
\lim_{k \to \infty} \lambda_{\nu} = \infty
\] (7.34)
because
\[
\sum_{\mu}^{\nu} \omega_{\mu} \leq \text{the number of irreducible paths (7.35)}
\]
for any \( k \) and
\[
\lim_{k \to \infty} \frac{\sum \lambda_{\nu} + \sum \omega_{\mu}}{} = \infty.
\] (7.36)

---

1) Needless to say that the number of irreducible loops as well as that of irreducible paths is finite in a finite network, even if we distinguish between loops having the same set of nodes and branches but with different nodes as the initial (and final) point. In the sequel we shall make such distinction as this.

- 37 -
Among the \( \nu \)'s for which (7.34) holds, there exists at least one \( \nu \) for which \( \xi < 1 \), since if we assumed the contrary we should have a contradiction to (7.30). Let us consider one of such loops, i.e., a loop for which

\[
\lim_{k \to \infty} \chi_\nu = \infty \quad (7.37)
\]

By virtue of the second formula of (7.37), we can find a \( k \) such that

\[
\chi_\nu > 5 \quad (7.38)
\]

for any \( \xi \). For such a \( k \), the \( \perp \)-composite of \( \gamma_\xi \)'s along the loop appears more than \( \xi \) times in (7.25). Therefore, there exists a sequence \( \{ h_1, h_2, \ldots, h_{2\xi} \} \) such that

\[
h_1 \geq N, \quad h_{2j} - h_{2j-1} = \text{the length of the loop} > 0, \quad h_{2\xi} \geq 0 \quad (7.39)
\]

and that

\[
\nu_a = \left( \sum_{\xi} \beta_\xi \sigma_{\xi} \right) \cdot \nu_a = \beta_\xi \nu_a + \gamma_\xi \nu_a \quad (7.40)
\]

around the loop.

It follows at once from (7.40) that, if we make \( \xi \) tend to infinity, we have

\[
u'_a = \lim_{k \to \infty} \nu_a = \lim_{\xi \to \infty} \nu_a = \lim_{\xi \to \infty} (\beta_\xi \nu_a + \gamma_\xi \nu_a) = \beta_\xi \nu_a + \gamma_\xi \nu_a = \sum_{\xi} \beta_\xi \sigma_{\xi} \cdot \nu_a \quad (7.41)
\]

around the loop.

which, however, implies the existence of conduction around the loop now being considered, contradicting the assumption that there exists no conduction loop with \( \beta_\xi < 1 \).

The proof of theorem 5 is thus completed.
Theorem 6. If a current-increasing step has finished with a finite $S$, in the next voltage-increasing step $u_1$ can certainly be increased.

Proof: If in the next voltage-increasing step $u_1 = \infty$ (or more precisely, $= O(M)$), the theorem holds because $\infty$ or $O(M)$ is, by definition, greater than any finite number we deal with. Therefore, let us assume $u_1 = o(M)$ (i.e., finite), and, furthermore, $u_1 = u_1$ (i.e., $u_1$ does not increase) contrarily to the conclusion of the theorem. (Note that $u_1 < u_1$ is excluded by Lemma in §5.) Then we can find a conduction route from the input node by virtue of the above Theorem 5. However all the voltages along that conduction route must be the same before and after the voltage-increasing step now under consideration, for, if $u_b = u_b$, $u_b = s_b u_a$ and $u_a \geq u_a$, then

$$u_b = \min_c s_b u_c \leq s_b u_a \leq s_b u_a = u_b = u_b,$$

and hence

$$u_a = u_a.$$

This means that the conduction route with regard to $u_1$ is conductive with regard also to $u_1$, which, however, contradicts the assumption that a current-increasing step has finished.

The following theorem is a by-product of the proof of Theorem 5.

Theorem 7. The only values that the $u_1$ obtained by a voltage-increasing step can take are

(i) $u_1 = \infty$,

(ii) $u_1 = \left( s_i \perp \ldots \perp s_i \right) (\text{where } s_i, s_i, \ldots, s_i, s_i, s_i \text{ are the nodes along an irreducible path from } s_i \text{ to } s_m)$,

and
(iii) \( u_L = (\sum_{\delta^r} \cdots \sum_{\delta^r} \sum_{\delta^r} u_{ck} \)

(\text{where } \delta^r, \delta^r, \cdots, \delta^r, \delta^r \text{ are the nodes along an irreducible path from } \delta^r \text{ to a certain node } \delta^r \text{ and } u_{ck} \text{ is the value})

which the voltage at \( \delta^r \) can take when a loop, containing \( \delta^r \) and having \( \beta < 1 \), is conductive.

The following theorem asserts, together with Theorem 4, that the \( S - u_L \) relation obtained by our method is the only admissible relation.

**Theorem 8.** Given a pair of current and voltage configurations which satisfies (2.1) - (2.4), then, with the current configuration fixed, the voltage configuration can be so varied that \( u_L \) may increase (decrease) only when \( S \) is the maximum (minimum) possible for the given voltage configuration; or, equivalently (contrapositive proposition!), with the voltage configuration fixed, \( S \) can be increased (decreased) only when \( u_L \) is the maximum (minimum) possible for the given current configuration.

**Proof:** If we perform a voltage-increasing step under the circumstances where \( S \) can be increased with the given voltages, we obtain the maximum possible \( u_L \) with the given current configuration fixed (see Theorem 2). By virtue of Theorem 5, we can increase the input current, namely \( S \), with regard to the new voltage configuration. Let us then increase \( S \) by a sufficiently small amount \( \varepsilon \) along the route. Then the increase in the total cost is, by (5.14),

\[ 1 \text{ It will be evident that this value is uniquely determined for a given node and a given loop. This value } u_{ck} \text{ will be called the proper node voltage at } \delta^r \text{ in regard to the } \nu \text{-th suction loop, and can be calculated from (7.41), or more explicitly by } \]

\[ u_{ck} = \frac{\delta_{\delta^r}}{1 - \beta} \quad (7.42) \]

where \( \delta_{\delta^r} \) is calculated by

\[ \delta_{\delta^r} = (1 - \beta \delta_{\delta^r}) \text{ around the } \nu \text{-th loop starting from } \delta^r \quad (7.43) \]

See also (9.4) in §9.
on the one hand. But, on the other, \( \delta \) could be increased with regard also to the old voltage configuration, so that we have

\[
\frac{z(\delta + \varepsilon)}{u_1} = \frac{z(\delta)}{u_1} \cdot \varepsilon.
\]  

(7.44)

Since \( z(\delta) \) is a single-valued function of \( \delta \) by definition, we have

\[
\frac{z(\delta + \varepsilon)}{u_1} = \frac{z(\delta)}{u_1}.
\]  

(7.45)

Hence we cannot increase \( u_1 \) under these circumstances.

The remaining part of the theorem will be obvious if we invert the sign of \( \delta \) (cf. the remark stated in the beginning of §3).

The theorems concerning the uniqueness of current and voltage configurations will also be found in a way analogous to that used in [2] and [4], but we shall not mention them here to avoid unnecessary complication.
The number of current-increasing steps and that of voltage-increasing steps included in the solution process of a problem of linear programming with weak graphical representation differ from each other by at most one, because the two kinds of steps are alternately used one after the other. From Theorem 7 in §7 it follows that the number of values \( u_1 \) can take is finite, and consequently, that the number of voltage-increasing step is finite. Hence the number of current-increasing step is also finite. Therefore, in order to prove the finiteness of the total number of minor substeps in a solution process, it suffices to prove the finiteness of the number of substeps included in a voltage-increasing step as well as of those included in a current-increasing step.

As regards a voltage-increasing step, the computational procedure proposed in §5.2 will terminate with a finite number of iterations when the \( u_1 \) to be obtained is of the form (ii) of Theorem 7, but will last without end (although convergence is guaranteed, of course) if it is of the form (iii) of Theorem 7. There is no problem in the former case, while we can find some finite computational procedures in the latter case as well, of which an example will be shown in §9.1. Thus, we may assume the number of necessary substeps in a voltage-increasing step to be finite.

As regards a current-increasing step, the problem is entirely analogous to that treated in [4] or [7]. We introduce a space of \( n \) dimensions with \( A S', A S^2, \ldots, A S'' \) as coordinates which represents incremental current configurations. The points in the space which correspond to the configurations satisfying (5.2) are confined within the intersection of a linear subspace (by the second and the fourth equation of (5.2)) and a convex region (by the third equation). It should be noted that the lower-dimensional convex region thus defined (which may be bounded

1) The convergence of \( v_1 \geq v_1 \geq \ldots \geq v_1 = v_1 = \cdots = u_1 \) (and consequently, the voltages along the conduction route) is ascertained by examining whether \( k^m = k \) or not.
or unbounded) has only a finite number of faces. The incremental input current \( \Delta S \) is defined as a linear function of coordinates of the points belonging to the convex region by the first equation of (5.2). It will be obvious that searching for a conduction route from the input node corresponds to searching for a \( \Delta S \)-increasing direction in the convex region, and assigning the maximum amount of incremental current along that conduction route, to going straight in that \( \Delta S \)-increasing direction until a boundary point of the convex region (i.e., a point belonging to a face) is reached. (The case where the infinite amount of incremental current can be assigned along the route is obviously compared to the case where the convex region is unbounded in the corresponding direction.)

The current-increasing step may be interpreted in geometrical terms of the convex region as follows where the statements in the parentheses will show the corresponding algorithms in the original form.

**First we start from the origin of the coordinate system \( \Delta S^x = 0 \), and search for a \( \Delta S \)-increasing direction in the convex region from the origin (we search for a conduction route from the input node\(^1\)). If no such direction can be found (no conduction route can be found\(^2\)), \( \Delta S \) is maximum there (the current-increasing step is completed). If a \( \Delta S \)-increasing direction is found, we go straight in that direction, reaching a point belonging to a face\(^3\) (some branches are brought in state \( B \) or \( D \) from another state), or finding the convex region unbounded in that direction (\( \Delta S \) can be increased infinitely). In the latter case the procedure stops, while in the former we search for a \( \Delta S \)-increasing direction confined within the face thus reached (we search for a conduction route regarding as

---

1) That this searching process is performed in a finite number of steps will be obvious from the flow diagram of Fig. 9.

2) The equivalence of the existence of a set of \( \Delta S^x \)'s satisfying (5.2) with \( \Delta S > 0 \) and of the fact that there is at least one conduction route such as shown in Fig. 8(a) or (b) will be evident from the condition that \( D^x = \text{sgn} \Delta S^x \) is the incidence matrix between the dimensions 0 and 1 of a finite (connected) network.

3) We mean an interior point of a face by "a point belonging to a face".
open-circuited the branches brought so far in state $B$ or $D$). If a $\Delta S$-increasing direction is found, we go straight in that direction, reaching a point belonging to a face of lower dimension, or finding the convex region unbounded in that direction. In these cases the procedure to follow is the same as the above, i.e.; we return to the place indicated just above by one asterisk *. If no $\Delta S$-increasing direction is found confined in the considered face, we call the face a "maximal face", and, regarding the point we now stand upon as the new origin, repeat the above procedure from the place indicated by two asterisks ** in the beginning of this paragraph (it is then ignored whether a branch has ever been brought in state $B$ or $D$ from another state).

Since the dimension of the face the moving point belongs to decreases monotonously until a maximal face is reached and the dimension of the convex region itself is finite; it requires only a finite number of steps to reach a maximal face. If a maximal face is reached, the dimension of the face the moving point belongs to will in general be increased. Moreover, since the value of $\Delta S$ is all the same at any point of a maximal face as will easily be proved) and $\Delta S$ steadily increases, no face can appear twice as the maximal face. Taking account of the finiteness of the number of faces of the convex region, we can thus conclude that a current-increasing step terminates after a finite number of substeps without fail.

1) The details about this fact (and the related facts) are entirely the same as those expounded in [7].
9. Discussions

9.1. A simplified method for determining voltage configurations. A voltage-increasing step may require infinite iteration under the form defined in §5.2. However, by virtue of Theorem 7 in §7, we can devise algorithms which require only a finite number of iterative substeps. The following is an example of such algorithms.

To begin with we put, as in (5.24),

\[ \psi = (v_b) \quad \text{and} \quad v_b = \begin{cases} 0 & \text{for } \tau^x_0 = \tau^x_m, \\ M & \text{for } \tau^x_0 \neq \tau^x_m, \end{cases} \]  

(9.1)

and calculate iteratively:

\[ \psi^k = B^k \cdot \psi \quad (k=0,1,2,\ldots) \]  

(9.2)

There is no problem if we have \( \psi^N = \psi \) for a finite \( N \). Otherwise we repeat the calculation until indication is observed that an element \( v_b \) is determined as

\[ v_b = (\perp \beta_{c_i}^{c_i^*})v_a \]  

(9.3)

where \( \perp \) is taken along a path containing a loop. If the feedback constant \( \beta \) around the loop thus found is greater than or equal to the unity, we proceed further with (9.2). If it is smaller than the unity, we calculate the proper node voltages around the loop by

\[ u_c = \frac{v_c - \beta v_c}{1 - \beta} \quad \left( \beta = \frac{v_c - v_c^{-1}}{v_c - v_c^{-1}} \right), \quad \text{or} \quad u_c = \frac{v_c v_c^{-1} v_c^2}{v_c + v_c^{-1} v_c^2} \]

where \( v_c = (\perp \beta_{c_i}^{c_i^*})v_c \) around the loop.

\[ (v_c : \text{arbitrary}) \]  

(9.4)

1) Note that the \( u_c \)'s thus defined satisfy the relation

\[ u_{c_i} = \beta_{c_i}^{c_i^*} u_{c_{i+1}} \]  

(9.4')

if \( \tau_{c_0}^x \) and \( \tau_{c_1}^x \) are two neighbouring nodes of the loop.

cf. (7.42) and (7.43)
for every node $\sigma^k_b$ belonging to the loop, put

$$v_b = \begin{cases} u_b & \text{if } \sigma^k_b \text{ belongs to the loop}, \\ v_b & \text{otherwise}, \end{cases} \quad (9.5)$$

and then return to (9.2), continuing until indication of the appearance of another loop is observed or we have $\hat{v}^N = \hat{v}$ with a finite $N$.

For the indication of the appearance of a loop we may take any property of the sequence $\{\hat{v}, \hat{\hat{v}}, \hat{v}, \ldots\}$ so far as it actually represents such a situation. For example, we may examine whether the voltage at a node ceases to decrease after iteration is made the number of times equal to the number of nodes or not. We can usually find more efficient means for individual problems. In these respects, see the examples to be shown in §10.
9.2. Use of $M$. We often made use of the symbol $M$ in the preceding sections, which may duly be regarded as corresponding to the extension of algebra $\mathcal{A}$ to the one such as explained below (cf. §4 of [2]).

(i) It has as its elements the linear forms

$$\mathcal{C} = xM + x', \quad (0 \leq x < \infty, \quad -\infty < x' < \infty)$$

and the symbol $\infty$.

In particular

$$OM + x'$$

is written simply as $x'$.

(ii) Two laws of composition $\perp$ and $\bowtie$ are defined as follows.

$$\mathcal{C}_1 \perp \mathcal{C}_2 = \left(\frac{xM + x'}{1} \right) \perp \left(\frac{xM + x'}{2}\right)$$

$$\text{def} = (x + x)M + (x' + x'), \quad \frac{1}{2} \quad \frac{1}{2}$$

$$\infty \perp \mathcal{C} = \mathcal{C} \perp \infty \text{ def } = \infty \quad (9.6)$$

$$\mathcal{C}_1 \bowtie \mathcal{C}_2 = \left(\frac{xM + x'}{1} \right) \bowtie \left(\frac{xM + x'}{2}\right)$$

$$\text{def} = \begin{cases} 
\mathcal{C}_1 & \text{if } x < \frac{x'}{2}, \text{ or if } x = \frac{x'}{2} \text{ and } x' < \frac{x'}{2}, \\
\mathcal{C}_2 & \text{if } x > \frac{x'}{2}, \text{ or if } x = \frac{x'}{2} \text{ and } x' \geq \frac{x'}{2},
\end{cases}$$

$$\infty \bowtie \mathcal{C} = \mathcal{C} \bowtie \infty \text{ def } = \mathcal{C} \quad (9.7)$$

(iii) Multiplication by a positive real number $\lambda$ is defined as the automorphism such that

$$\mathcal{C} = xM + x' \rightarrow \lambda \mathcal{C} = (\lambda x)M + (\lambda x'), \quad \infty \rightarrow \infty \quad (9.8)$$

(iv) The algebra corresponding to $\mathcal{A}'$ (see §4 of [2]) is naturally obtained by defining the law of operation of the operator $\beta(c,d,e)$ on $\mathcal{C}$ or $\infty$ as in (9.9).
\[ \beta(c,d,e) = \frac{c}{d} \quad (9.9) \]

(v) In order to make clear the concept of limit in this case, we define the norm \( \| \cdot \| \) of \( \mathcal{V} \) by

\[
\begin{align*}
\| \mathcal{V} \| &= \| xM + x' \| \quad \text{def}= x^2 + x'^2, \\
\| \infty \| &= \infty.
\end{align*}
\]  

(9.10)

Then we define

\[
\lim_{k \to \infty} \mathcal{V} = \mathcal{V} \quad (\mathcal{V} \neq \infty) \quad (9.11)
\]

by the stipulation\(^2\):

for any \( \varepsilon > 0 \) there exists an \( N \) such that, if \( k > N \), an element \( \mathcal{V} \) whose norm is smaller than \( \varepsilon \) (\( \| \mathcal{V} \| < \varepsilon \)) exists satisfying

\[
\mathcal{V} = \mathcal{V} \perp \mathcal{V}, \quad (9.12)
\]

and

\[
\lim_{k \to \infty} \mathcal{V} = \infty
\]

by the stipulation\(^2\):

for any \( \varepsilon > 0 \) there exists an \( N \) such that, if \( k > N \),

\[
\| \mathcal{V} \| > \varepsilon, \quad (9.13)
\]

where some of \( \mathcal{V} \)'s may be \( \infty \) in (9.11) and in (9.13).

\(^1\) There are many alternative definitions, such as \( |x| + |x'| \), \( 4x^2 + |x'| \), \( |x| + 7x^6 \), etc. They will as well serve for our purpose as (9.10).

\(^2\) Of course, \( \varepsilon \) is a real number and \( N \), a positive integer.
9.3. A special case for which the solution process can extremely be simplified. Let us consider the problems of the following type as a special case of the problems studied in the preceding sections. They are the problems of linear programming with weak graphical representation in which

\[ e_\infty = 0 \]  \hspace{1cm} (9.14)^1

except for the branches terminating at the output node \( \sigma_m \). The problems of this type will often occur in practice. The problem taken up in §1 as a simple example can be transformed into this type.\(^2\) Of course, this problem can be solved by the method so far explained, but we shall see that the solution process, if applied to a problem of this type, is extremely simplified.

The most remarkable features of the problems of this type may be summarized in the following theorem.

Theorem 9. If (9.14) holds for a problem, no loop having feedback constant smaller than unity becomes conductive except in the first current-increasing step.

Proof: The proper node voltage \( u_\infty \) at node \( \sigma_k \) in regard to a loop which does not contain the output node \( \sigma_m \) is always 0, as will easily be seen from (7.42) or (7.43) or (9.4). Therefore, if a suction loop appears after a voltage-increasing step, we have, from (iii) of Theorem 7,

\[ u_i = (e_{c_i} \perp \ldots \perp e_{c_{k-1}}) u_\infty \]

\[ = 0, \]  \hspace{1cm} (9.15)

because \( u_\infty = 0 \) and no voltage source \( e_\infty \neq 0 \) exists along the path \( \sigma^i_k, \sigma^2_k, \ldots, \sigma_m \). But this does not occur except in the first voltage-increasing and the first current-increasing step.

\(^1\) We may assume without loss of generality that \( e_\infty \) is negative if not null.

\(^2\) For this transformation, see Fig.13 in Example 3 of §10.

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By virtue of Theorem 9, the cumbersome procedures owing to the appearance of suction loops can be dispensed with. The number of iterations in a voltage-increasing step, as well as that in a current-increasing step, is finite in this case, nor is the use of the symbol \( M \) necessary. \( M \) may be replaced simply by \( \infty \), the related algebra being again reduced to \( \mathbb{A} \) with

\[
x_\infty + x' = \infty \quad (x \neq \infty) \quad (9.16)
\]

Indeed, the solution process is simplified as will be shown below in (i) and (ii).

(i) Current-increasing steps: — A current-increasing step is performed in the same way as in the general case described in \( \S 5.1 \). But we need not expect the appearance of suction loops except in the first current-increasing step. Therefore, computational labour required in the current-increasing steps is saved to some extent and is nearly equal to that required in the current-increasing steps of a transportation problem.

(ii) Voltage-increasing steps: — Much simplification is made in these steps. In a voltage-increasing step, we put according to (5.24)

\[
\begin{align*}
\dot{\nu} &= (\dot{\nu}_b) : \\
\nu_b &= \begin{cases} 
0 & \text{for } \tau^*_b = \tau^*_m, \\
\infty \text{ (or M)} & \text{otherwise.}
\end{cases}
\end{align*}
\]

Then \( \ddot{\nu} = B \cdot \dot{\nu} \) is

\[
\begin{align*}
\ddot{\nu} &= (\ddot{\nu}_b) : \\
\nu_b &= \begin{cases} 
0 & \text{if } \tau^*_b = \tau^*_m, \\
e^* & \text{if } \sigma^*_b = \tau^*_b - \sigma^*_m \text{ and branch is not saturated, i.e. in state } \mathcal{C}, \mathcal{D} \text{ or } \mathcal{E}, \\
\infty & \text{otherwise,}
\end{cases}
\end{align*}
\]

for, the only branches having non-vanishing \( e^* \) are such that \( \sigma^*_b = \tau^*_b - \sigma^*_m \). Since \( \dot{\nu}_m \) is always 0 and

---

1) Note that we have assumed that \( \mathcal{D}^*_m \) is either negative or null. cf. footnote 1) on p. 49.
we have, for \( k \geq 1 \),
\[
\tilde{v}_{k}^{a} = \tilde{t}_{k}^{a} \left( \begin{array}{c}
\tilde{v}_{k}^{a} \\
\tilde{v}_{k}^{b}
\end{array} \right) = \left\{ \tilde{t}_{k}^{a} \left( \begin{array}{c}
\tilde{v}_{k}^{a} \\
\tilde{v}_{k}^{b}
\end{array} \right) \right\} T \begin{array}{c}
\theta_{k}^{a} \\
\theta_{k}^{b}
\end{array} \tilde{u}_{k}^{c}
\]
\[
= \left\{ \tilde{t}_{k}^{a} \left( \begin{array}{c}
\tilde{v}_{k}^{a} \\
\tilde{v}_{k}^{b}
\end{array} \right) \right\} T \begin{array}{c}
\theta_{k}^{a} \\
\theta_{k}^{b}
\end{array} \tilde{u}_{k}^{c}
\]
\[
= \tilde{t}_{k}^{a} \left( \begin{array}{c}
\tilde{v}_{k}^{a} \\
\tilde{v}_{k}^{b}
\end{array} \right) \tilde{u}_{k}^{c}
\]

because of the relations
\[
\tilde{t}_{k}^{a} \left( \begin{array}{c}
\tilde{v}_{k}^{a} \\
\tilde{v}_{k}^{b}
\end{array} \right) = \left\{ \tilde{t}_{k}^{a} \left( \begin{array}{c}
\tilde{v}_{k}^{a} \\
\tilde{v}_{k}^{b}
\end{array} \right) \right\} T \begin{array}{c}
\theta_{k}^{a} \\
\theta_{k}^{b}
\end{array} \tilde{u}_{k}^{c} = \tilde{t}_{k}^{a} \left( \begin{array}{c}
\tilde{v}_{k}^{a} \\
\tilde{v}_{k}^{b}
\end{array} \right) \tilde{u}_{k}^{c}
\]
\[
\tilde{v}_{k}^{a} = \tilde{v}_{k}^{b} \tilde{u}_{k}^{c}
\]

we can hereafter delete the \( m \)-th element from \( \tilde{v} \) and the corresponding row and column from \( B \), denoting the \( \tilde{v} \) and \( B \) thus reduced by \( \tilde{v} = (\tilde{\tilde{v}}) \), and \( \tilde{B} = (\tilde{\tilde{B}}) \), \((b=1, \ldots, m-1; k=2,3,\ldots)\). (5.25) is then reduced to
\[
\tilde{v}^{a} = \tilde{B}^{-1} \tilde{v}^{a}
\]
\[
(9.19)
\]

where \( \tilde{\tilde{B}} = (\theta_{a}^{b}) \), \((a,b=1,\ldots,m-1)\). All the elements of \( \tilde{B} \) is, however, composed of such operators as
\[
\varphi(0,d,e) \quad \text{and} \quad \varphi(\infty,d,e),
\]
\[
\varphi(0,d,e) x = (e/d)x,
\]
\[
\varphi(\infty,d,e) x = \infty.
\]

The operator \( \varphi(0,d,e) \) represents nothing but the multiplication by \( e/d \). As is easily seen, such \( \varphi \)'s constitute a subalgebra of algebra \( A' \) and is isomorphic to algebra \( A \) (if it is formally assumed that \( \infty \times x = \infty \) for any \( x \)) by the correspondence:

\[
\text{subalgebra of } A' \quad : \quad A,
\]
\[
\varphi(0,d,e)(\text{or } e/d) \quad \longleftrightarrow \quad \log(e/d),
\]

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\[ \beta (\infty, d, e) (\text{or } \infty) \leftrightarrow \infty, \]
\[ \perp (X) \leftrightarrow \perp (+), \]
\[ T (\text{min}) \leftrightarrow T (\text{min}). \]

Hence in a voltage increasing step, first we define \( \tilde{\nu} \) by (9.18) and then calculate
\[ \tilde{\nu} = \lim_{k \to \infty} \tilde{\nu} = \lim_{k \to \infty} (\tilde{B} \ast \tilde{B} \ast \ldots \ast \tilde{B}) \tilde{\nu}, \quad (9.21) \]
using the above-mentioned subalgebra of \( A' \). \( \tilde{\nu} \) will be obtained with a finite \( k \). Moreover if we calculate \( \tilde{\nu} \) by the logarithmic scale, then calculation will become quite the same as that in a transportation-network problem. Hence the labour required in a voltage-increasing step is quite the same as that required in a voltage-increasing step of a transportation problem.

Thus in the case of a problem of linear programming with weak graphical representation of the type discussed in this section, the required labour is nearly equal to that of a problem of linear programming with strong graphical representation, i.e., a transportation problem. The solution process may be performed in a way exactly analogous to that for a transportation problem. 1)

---

1) We can make use of the programming of a digital computer for the transportation problems also for the problems of the type discussed in this section with the slight modification of the current-increasing steps and no modification of the voltage-increasing steps.
10. Examples

In order to illustrate the general method proposed in the preceding sections for linear-programming problems with weak graphical representation we shall show four examples in this section. The first of them is very simple but will serve as a model problem by which to understand the general method. The second is the example to show how things are simplified when the underlying network structure is of the Hitchcock type. The third is the problem which was used in §1 as the introductory example, and we shall see this type of problems belong to the type mentioned in §9.3. In the last example a fairly complicated and practically significant problem will be attacked.

Example 1. Let us solve the problem shown in Fig. 10. Following exactly the solution procedure presented in §5-§7 and utilizing some minor tricks in the voltage increasing.
steps (see (21.6) of [2] and also §4.2 of [13]) we have the
following tableaux. The form of these tableaux is entirely
the same as explained in [2] and [13] save that each element
in the matrix B represents the triple \((\pm e_x ; D_x, D_x')\)
of
\(\beta_v'' = \beta (\pm e_x ; D_x, D_x')\) and that each element of C repres-
sents the triple obtained by replacing the member \(\pm e_x\) of
the corresponding element of B by the branch capacity for
incremental current.

\[
\begin{array}{c|ccccc}
\text{a} & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & (0,1,1)(50,2,1/5) & \infty & (1,1/2,1) & \infty \\
B = 2 & (\infty,1,1/2)(0,1,1)(8,2,1/2)(\infty,1/5,1/3) & \infty \\
3 & \infty & (\infty,1/2,2)(0,1,1)(1,1,1/2)(12,2,1/4) \\
4 & (\infty,1,1/2)(1,1/3,1/5)(\infty,1/2,1)(0,1,1)(1,2,1/3) \\
5 & \infty & \infty & (\infty,1/4,2)(\infty,1/3,2)(0,1,1) \\
\end{array}
\]

(We write simply \(\infty\) for the
entries where there is no
corresponding branch.)

\[
\begin{array}{ccccc}
\text{c} & 1 & 2 & 3 & 4 & 5 \\
\hline
\nu & \begin{bmatrix} M & M & M & M & 0 \end{bmatrix} \\
\nu' & \begin{bmatrix} 3 & 4 & 5/16 & 1/4 & 1/2 & 0 \end{bmatrix} \\
\nu'' & = \nu = \nu' \\
\end{array}
\]

\(\nu_1 = 3\)

---

1) The calculation is proceeded as follows.
\(\tilde{v}_4 : \beta(1,2,1/3)\tilde{v}_5 = \beta(1,2,1/3) \cdot 0 = 1/2\),
\(\tilde{v}_3 : [\beta(12,2,1/4) \cdot \tilde{v}_5]^{\top} \beta(1,1,1/2) \tilde{v}_4 = (6) \top (5/4) = 1/4\),
\(\tilde{v}_2 : \beta(8,2,1/2) \cdot \tilde{v}_3 = 4/16\),
\(\tilde{v}_1 : [\beta(1,1/2,1) \cdot \tilde{v}_4]^{\top} \beta(50,2,1/5) \cdot \tilde{v}_2 = (3) \top (4069/160) = 3\).

(continued on the next page)
\( b \)
\[
\begin{array}{cccccc}
  & 1 & 2 & 3 & 4 & 5 \\
\hline
 a & & & & & \\
 1 & 0 & (10, 2, \frac{1}{5}) & 0 & (2, 2, 1) & 0 \\
 2 & (0, \frac{1}{5}, 2) & 0 & (1, 2, \frac{1}{5}) & (0, \frac{1}{5}, \frac{1}{3}) & 0 \\
 3 & 0 & (0, \frac{1}{2}, 2) & 0 & (1, 1, \frac{1}{2}) & (1, 2, \frac{1}{4}) \\
 4 & (0, 1, \frac{1}{2}) & (1, \frac{1}{3}, \frac{1}{5}) & (0, \frac{1}{2}, 1) & 0 & (1, 2, \frac{1}{5}) \\
 5 & 0 & 0 & (0, \frac{1}{4}, 2) & (0, \frac{1}{5}, 2) & 0 \\
\end{array}
\]
\[ \Delta S = 1 \]
\[ u_1 \Delta S = 3 \]

(We write simply \( 0 \) for the entries where there is no corresponding branch. The underlines indicate conductive branches.)

\[ \langle V_2 & C_2 \rangle \]

\[
\begin{array}{cccccc}
  & 1 & 2 & 3 & 4 & 5 \\
\hline
 a & & & & & \\
 1 & (0, 1, 1) & (5, 2, \frac{1}{5}) & \cdots & (\infty, \frac{1}{2}, 1) & \cdots \\
 2 & (\infty, \frac{1}{5}, 2) & (0, 1, 1) & (8, 2, \frac{1}{2}) & (\infty, \frac{1}{5}, \frac{1}{3}) & \cdots \\
 3 & \cdots & (\infty, \frac{1}{2}, 2) & (0, 1, 1) & (1, 1, \frac{1}{2}) & (12, 2, \frac{1}{4}) \\
 4 & (1, 1, \frac{1}{2}) & (1, \frac{1}{3}, \frac{1}{5}) & (\infty, \frac{1}{2}, 1) & (0, 1, 1) & (\infty, 2, \frac{1}{3}) \\
 5 & \cdots & \cdots & (\infty, \frac{1}{4}, 2) & (-1, \frac{1}{3}, 2) & (0, 1, 1) \\
\end{array}
\]

(continued from the previous page)

Namely, it is effective to calculate the components in descending order from \( v_{m-1} \), replacing successively the old values by new one, as is made in Gauss-Seidel method for the solution of linear simultaneous equations. The reason is quite obvious from the physical meaning of \( v_b \). Obviously this \( v_b \) can be obtained by calculating \( v = B v_b \) repeatedly. The conduction route in current increasing step \( \langle C_1 \rangle \) below will be found directly from above calculation procedures of \( v_b \).

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\[ \mathbf{v}^r = \begin{bmatrix} M & M & M & M & 0 \\ \frac{511}{20} & 11 & 6 & M & 0 \end{bmatrix} \]
\[ \mathbf{v}^i = \begin{bmatrix} \frac{20403}{800} & 403 & 83 & 63 & 0 \\ 80 & 80 & 20 & 10 & 0 \end{bmatrix} \]
\[ \beta \nu = \left( \frac{1}{2} \times \frac{1}{2} \right) \times \left( \frac{1}{5} \times 3 \right) \times \left( \frac{1}{2} \times 1 \right) \]
\[ \lambda - \beta \nu \delta = \frac{3}{40} \left( \frac{83}{37} - \frac{3}{40} \right) = \frac{148}{37} = 4, \]
\[ \text{put } \frac{3}{2} \nu_3 = 4, \frac{7}{2} \nu_2 = 5, \frac{3}{2} \nu_4 = 6. \]
\[ \mathbf{v}^3 = \begin{bmatrix} 20403 & 5 & 4 & 6 & 0 \\ 800 & 51 & 5 & 4 & 6 \end{bmatrix} \]
\[ \mathbf{v}^4 = \mathbf{v}, \quad \frac{1}{2} \mathbf{v}^4 = 51/2 \]

\[ C = \begin{array}{cccccc}
1 & 0 & (0, 1, 1) & 0 & (0, 1, 1) & 0 \\
2 & (0, 1, 1) & 0 & (0, 1, 1) & 0 & (0, 1, 1) \\
3 & 0 & (0, 1, 2) & 0 & (1, 1, 2) & (1, 2, 1) \\
4 & (2, 1, 1) & (1, 3, 2) & (0, 1, 1) & 0 & (0, 2, 1) \\
5 & 0 & 0 & (0, 1, 2) & (1, 3, 2) & 0 \\
\end{array} \]

Here a loop with $\beta < 1$ has appeared. The values of $\nu$'s attached with superscripts $\alpha$, $\beta$, $\gamma$, and $\delta$ have been determined from the following sequence $\delta \rightarrow \gamma \rightarrow \beta \rightarrow \alpha$. This means a loop passing through nodes $\gamma_3(\alpha) \rightarrow \gamma_4(\beta) \rightarrow \gamma_2(\gamma) \rightarrow \gamma_3(\alpha)$ is a suction loop. It is detected by inspecting the connection between the suffixes of the preceding and new

(continued on the next page)
values of $v_1$. Namely, in this case 2(3), 3(4), 4(2), where each number preceding the parentheses one is the suffix of the new value and the latter is the suffix of the previous $v$ from which the new value is induced.
\( B = \begin{bmatrix}
1 & (0,1,1)(\infty,2,\frac{1}{5}) & (\infty,\frac{1}{2},1) \\
2 & (-50,\frac{1}{5},2)(0,1,1)(\infty,2,\frac{1}{2})(\infty,\frac{1}{3},\frac{1}{3}) \\
3 & (\infty,-8,\frac{1}{2},2)(0,1,1)(1,1,\frac{1}{2})(12,2,\frac{1}{4}) \\
4 & (-1,1,\frac{1}{2})(1,\frac{1}{3},\frac{1}{5})(-1,\frac{1}{2},1)(0,1,1)(\infty,2,\frac{1}{3}) \\
5 & (\infty,\infty,(-12,\frac{1}{4},2)(-1,\frac{1}{3},2)(0,1,1)
\end{bmatrix}
\]

\[
\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} M \\ M \\ M \\ M \\ M \end{bmatrix}
\]

The solution process has come to an end. For example, the maximum amount of input current is \( \frac{3}{4} S + \frac{2}{4} S + \frac{1}{4} S = 21 \), the corresponding current configuration is as shown below.

\[
\begin{bmatrix}
21 & 1 & 2 & 3 & 4 \\
1 & 0 & 10 & 0 & 2 \\
2 & 0 & 0 & 1 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & \frac{1}{4} \\
4 & 0 & 0 & 0 & 0 & 1 \\
5 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

and the total cost for this configuration (which is minimum with the input current fixed to 21) is

\[
3 \cdot \frac{1887}{4} + \frac{157}{4} + 50 \times 10 + 2 \times 1 + 1 \times 8 + \frac{1}{4} \times 12 + 1 \times 1 = 514.
\]

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Example 2. Three machines are available for the production of two kinds of parts. The time and cost required for the production of unit amount of each part vary with the machine by which it is produced. The numbers of parts to be produced and the available machine-hours are also given. Obtain the schedule minimizing the total production cost. The data are

<table>
<thead>
<tr>
<th>Production hours per unit per machine ($e_{ij}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Machine $j$</td>
</tr>
<tr>
<td>Part $i$</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
</tbody>
</table>

| Available machine hours $b_j$ | 12 | 5  | 8  |

<table>
<thead>
<tr>
<th>Production cost per unit per machine ($c_{ij}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Machine $j$</td>
</tr>
<tr>
<td>Part $i$</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Number of parts required $a_i$</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3</td>
</tr>
</tbody>
</table>

Hence the problem is to obtain the set of $x_{ij}$'s ($x_{ij}$ = amount of part $i$ to be produced by machine $j$) which minimizes

$$z = \sum_{i=1}^{3} \sum_{j=1}^{2} c_{ij} x_{ij}$$  \hspace{1cm} (10.2)

under the conditions:

$$\sum_{j=1}^{2} x_{ij} = a_i, \hspace{1cm} (i=1,2)$$  \hspace{1cm} (10.3)

$$\sum_{i=1}^{3} e_{ij} x_{ij} \leq b_j, \hspace{1cm} (j=1,2,3)$$  \hspace{1cm} (10.4)

and

$$x_{ij} \geq 0 \hspace{1cm} (i=1,2; \ j=1,2,3).$$  \hspace{1cm} (10.5)
The network representing this problem is shown in Fig.11, where we want to maximize with as small expense as possible.

We use the tableaux of the form (10.6) for computation.

$$b_j - \sum_{i} e_{ij}x_{ij}$$

$$u_i = v_i \cdots v_i \cdots v_i$$

$$c_{ij}/x_{ij} ; e_{ij} a_i$$

(if $x_{ij}=0$, put simply $c_{ij}x_{ij}$)

$$\sum_{i} x_{ij}$$

$$u_i = v_i \cdots v_i \cdots v_i$$

$$\sum \Delta z = u \Delta \lambda$$
In (10.6), \( x_{ij} \)'s represent the schedule already assigned, \( b_j - \sum_i e_{ij} x_{ij} \) the machine-hour still available for machine \( j \), \( a_i - \sum_j x_{ij} \) the amount of part \( i \) to be produced further. The circles in the tableau indicate that the corresponding branches are conductive (i.e. \( u_b = \varepsilon \un u_a \)). (cf. [2] and [13] for the details of tableaux.) The node voltages \( \nu_j \)'s are determined by:

\[
\nu_j = \begin{cases} 
0, & \text{if } b_j - \sum_i e_{ij} x_{ij} \neq 0 \\
M, & \text{otherwise}
\end{cases},
\]

(10.7)

\[
\nu_j = \min_i \frac{1}{e_{ij}} (\nu_i - c_{ij}) \quad (k=1,2,\ldots),
\]

(10.8)

where \( \min \) means that the minimum value is taken among those \( \frac{1}{e_{ij}} (\nu_i - c_{ij}) \)'s to which the corresponding \( x_{ij} \)'s are not null, and \( \nu_i \)'s by

\[
\nu_i = \min_j \frac{1}{e_{ij}} (e_{ij} \nu_j + c_{ij}) \quad (k=1,2,\ldots).
\]

(10.9)

The solution process proceeds as follows.

\(< V_1 & C_1 > \)

\[
\begin{array}{cccc}
12 & 5 & 8 \\
1 & 3 ; 1 & 6 ; 2 & 13 \\
2 & 2 ; 4 & 3 ; 2 & 14 ; 10 \\
1 & 0 & 0 & 0 \\
\end{array}
\]

\( \Delta z = 5 \)

\( \sum \Delta z = 5 \)

\(< V_2 & C_2 > \)

\[
\begin{array}{cccc}
12 & 0 & 8 \\
3 & 3 ; 1 & 1/5 ; 1 & 6 ; 2 & 8 \\
2 & 2 ; 4 & 3 ; 2 & 14 ; 10 \\
2 & 0 & M & 0 \\
\end{array}
\]

\( \Delta z = 6 \)

\( \sum \Delta z = 11 \)
Here it is noted that \( \frac{3}{\lambda} = 5.75 \) indexed by \( \ell \) is
determined around a loop, i.e. it is determined by
\( 3 + 1 \times 2.75 \) (2.75 is indexed by \( \delta \) in the tableau),
2.75 is determined by \( (13-2)/4 \) (13 is indexed by \( \gamma \)),
13 is obtained by \( 3 + 2 \times 5 \) (5 is indexed by \( \beta \)),
and finally 5 is determined by \( (6-1)/5 \) (6 is indexed
by \( \alpha \)). Hence we see a loop with \( \beta < 1 \)
appear (see Fig.12). In such a case, we calculate the proper node voltages of the loop. The
feed back constant \( \beta_a \) of this loop is,
\[
\beta_a = 1 \times \frac{1}{4} \times 2 \times 1 = \frac{1}{2},
\]
and the proper node voltage at \( \alpha \) is
\[
\mathcal{V}_{\alpha} = \frac{\delta}{1-\beta_a} = \frac{1}{1-\frac{1}{2}} \left( 3 - \frac{1}{4} \times 2 + \frac{1}{4} \times 3 - \frac{1}{4} \times 2 \times 1 \right)
= 2 \left( 2 \frac{3}{4} \right) = 5.5.
\]
In a similar way, we obtain
\[
\hat{\beta} = 4.5, \quad \hat{\gamma} = 12
\]
and \( \hat{\delta} = 2.75 \).
Take these values for $v_1$'s and $v_1$'s, and continue the iteration (10.8) - (10.9). We see, however, the iterative process have already converged. Hence, proceed to the current-increasing step $<C_3>$. In this case the incremental current is absorbed in the suction loop. The maximum incremental current $\Delta S$ is, by (5.12),

$$\Delta S = \min \left\{ (1-\beta_1)5 \times 2, (1-\beta_2)3 \times 4 \right\} = 5$$

Thus, the assignment of parts to machines are complete. The resultant schedule is shown in (10.10).

<table>
<thead>
<tr>
<th>Machine j</th>
<th>Part i</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>Unassigned amounts</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>3</td>
<td></td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0.5</td>
<td>2.5</td>
<td></td>
<td></td>
<td>0</td>
</tr>
</tbody>
</table>

Unused machine-hours

0 0 2

The corresponding total cost is 56.5, the unused machine-hours are 2 for machine 3.
Example 3. The table of data for the problem mentioned in §1, is reproduced below.

<table>
<thead>
<tr>
<th>Production hours per unit per machine ( (e_{ij}) )</th>
<th>Numbers of parts required ( a_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Machine ( j )</td>
<td>1</td>
</tr>
<tr>
<td>Part ( i )</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>6</td>
</tr>
</tbody>
</table>

| Available machine hours \( b_j \) | 80 | 30 | 160 |

The problem is to minimize the total machine-hours. First, it is noticed that the network model of Fig.1(§1) can be transformed into the equivalent network shown in Fig.13, for
which (9.14) in §9 holds. Hence the problem is of the type discussed in §9.3 and can be solved by a simple procedure.

We shall use the following tableaux for computation.

\[
\begin{array}{c|c|c|c}
\hline
b_j - \sum_{i} e_{ij} x_{ij} & \sum_{i} e_{ij} x_{ij} & a_i - \sum_{j} x_{ij} \\
\hline
\end{array}
\]

\[
(\text{if } x_{ij} = 0, \text{ then put simply } e_{ij})
\]

\[
z = u \Delta S
\]

The circles in the tableaux indicate the conductive branches. The voltages \(v_j\)'s are determined by

\[
\frac{1}{v_j} = \begin{cases} 
1 & \text{if } b_j - \sum e_{ij} x_{ij} = 0, \\
\infty & \text{otherwise}, 
\end{cases} \quad (10.12)
\]

\[
v_j = \min_{i} \frac{k-l}{v_i}, \quad (k=2,3,\cdots) \quad (10.13)
\]

where \(\min\) is the operation of taking the minimum of such \(v_i\)'s as correspond to a non-vanishing \(x_{ij}\). \(v_i\)'s are determined by

\[
v_i = \min_{j} e_{ij} v_j. \quad (10.14)
\]
The solution proceeds as follows.

### \( V_1 \& C_1 \)

<table>
<thead>
<tr>
<th></th>
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<th>1</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>30</td>
<td>30</td>
<td>30</td>
<td>30</td>
<td>30</td>
</tr>
</tbody>
</table>

\( \Delta z = 100 \)

\( \Sigma \Delta z = 100 \)

### \( V_2 \& C_2 \)

<table>
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<tr>
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<th>1</th>
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</thead>
<tbody>
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<td>4</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>1/30</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>30</td>
<td>30</td>
<td>30</td>
<td>30</td>
<td>30</td>
</tr>
</tbody>
</table>

\( \Delta z = 150 \)

\( \Sigma \Delta z = 250 \)
Thus the whole amount of parts has been assigned to the machines and the problem is completely solved. It should be noted that the required labour is of quite the same order as that required for solving a transportation problem of the same size. The solution $x_{ij}$'s are

<table>
<thead>
<tr>
<th>Machine j</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>$x_{ij}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parts i</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>10</td>
<td>10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>40</td>
<td>40</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>30</td>
<td>30</td>
<td>60</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>20</td>
<td>20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>20</td>
<td>10</td>
<td>30</td>
<td></td>
</tr>
</tbody>
</table>

and the corresponding total cost is 270.
Example 4. As the last examples we take up the problem of providing thermal power stations with coal from a number of mines, which was treated by J. Abadie in [14]. The given conditions are as follows:

(i) There are $m-1$ mines, the $i$-th of which produces $Q_i$ of coal ($i=1,2,\ldots,m-1$);

(ii) There are $n$ thermal power stations, the $j$-th of which is capable of generating not more than $E_j$ of electric energy per year ($j=1,2,\ldots,n$);

(iii) The total amount of demand of electric energy per year is $E_0$;

(iv) The efficiency of the $j$-th power station is such that it generates $h_j$ of electric energy from the unit amount of coal;

(v) The amount of coal produced in the domestic mines is not sufficient to meet the demand, so that we must import a certain amount of coal (which amount we denote by $Q_m$ assuming the $m$-th imaginary mine) but the amount is not limited in advance;

(vi) It costs as much as $c_{ij}$ ($i=1,2,\ldots,m$; $j=1,2,\ldots,n$) to provide the $j$-th station with the unit amount of coal from the $i$-th mine (including the imaginary $m$-th mine);

(vii) There exists technical or some other restriction which makes it impossible to provide the $j$-th station with coal from the $i$-th mine for some pairs $(i,j)$; such restriction as this can however be replaced by the condition that $c_{ij} = \infty$ for such pairs.

The problem is to obtain such a manner of delivering coal to the stations as minimizes the total cost for provision.

The above conditions are formulated as shown in (10.16) through (10.20) if we denote by $x_{ij}$ the amount of coal from the $i$-th mine to the $j$-th station. 1)

\[ 1) \text{As a matter of course, } x_{ij} \geq 0. \]
\[
(\text{i}) \quad \sum_{j=1}^{n} x_{ij} \leq Q_i \quad (i=1,2,\ldots,m-1), \quad (10.16)
\]

\[
(\text{v}) \quad Q_m \stackrel{\text{def}}{=} \sum_{j=1}^{n} x_{mj} = \text{arbitrary}, \quad (10.17)
\]

\[
(\text{ii}(\text{iv})) \quad \sum_{j=1}^{n} x_{ij} \leq R_j \stackrel{\text{def}}{=} E_j / h_j \quad (j=1,2,\ldots,n), \quad (10.18)
\]

\[
(\text{iii}) \quad \sum_{j=1}^{n} h_j \left( \sum_{i=1}^{m} x_{ij} \right) = E_0 , \quad (10.19)
\]

\[
(\text{iv}) \quad f \stackrel{\text{def}}{=} \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} = \text{total cost.} \quad (10.20)
\]

Abadie solved the problem of minimizing \( f \) under the additional condition that the inequalities in (10.16) should be the equalities (i.e., all the coal produced in the domestic mines should be consumed). We shall call such a problem "Problem B" and solve it later, because it seems more natural to assume the problem as that of first minimizing \( Q_m \) (in (10.17)) and then minimizing \( f \) (in (10.20)) (i.e., the amount of coal to be imported should be as small as possible), calling the latter problem "Problem A".

First, let us solve "Problem A". It is easily seen that this problem is equivalent to the minimization of

\[
z = f - \mathcal{M} Q_m = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} - \mathcal{M} \sum_{j=1}^{n} x_{mj} \quad (10.21)
\]

under the conditions (10.16), (10.18) and (10.19). As has so far been explained, such a problem as this can be converted into the linear programming with weak graphical representation, to which the corresponding network structure is as shown in Fig. 14. We shall use, in the following computation, the abridged tableau such as (10.22), of which the meaning will not be difficult to understand without
explanation.

The example which Abadie treated in [14] is, in the form of our abridged tableau, such as shown in (10.23), where $x_{ij} = 0$ for all $i$ and $j$. We shall take advantage of the fact that a suction loop in this kind of problem, if any exists, contains always the input node. Since we know that the production of coal in the domestic mines is not sufficient by any means\(^1\), the optimum solution must be such that all the inequalities in (10.16) are reduced to equalities. Therefore, the optimum solution for the given problem will coincide with that for the reduced problem which is

\[ s \text{(to be increased up to } E_0) \]

\[ j=1 \]

\[ i=2 \]

\[ n \]

\[ \sum_{t=1}^{n} x_{ij} \]

\[ \text{nodes corresponding to power stations} \]

\[ \text{nodes corresponding to mines} \]

\[ \text{Fig. 14} \]

\(^1\) This is ascertained again in the course of the following calculation.
specified by the tableau (10.24) instead of (10.23).

<table>
<thead>
<tr>
<th>45</th>
<th>55</th>
<th>67</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>∞</td>
<td>0</td>
</tr>
<tr>
<td>∞</td>
<td>4</td>
<td>∞</td>
</tr>
<tr>
<td>11</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>14</td>
<td>6</td>
<td>8</td>
</tr>
</tbody>
</table>

Through this modification, the total cost is diminished by

\[ 17 \times 45 + 18 \times 55 + 21 \times 67 = 3162. \]  

(10.25)

The solution process proceeds as follows.

\[ \langle V_1 & C_1 \rangle \]

\[ \begin{array}{ccc|c|c|c}
45 & 55 & 67 & 17 & 52 & 45 \\
0 & 0 & 0 & 0 & 30 & (17,0) & 4 \\
1 & 6 & 1 & 1 & 29 & (22,0) & 5 \\
0 & 7 & ∞ & 4 & 32 & (52,0) & 5 \\
4 & ∞ & 4 & ∞ & 36 & (41,0) & 6 \\
5 & 11 & 5 & 5 & 38 & (70,0) & 8 \\
6 & 14 & 6 & 8 & 37 & (50,0) & 9 \\
\hline
0 & 0 & 0 & 0 & 1451 & & \\
\end{array} \]

1) The first three columns for c_{ij} are obtained from the corresponding columns of (10.23) by subtracting, respectively, 17, 18 and 21.
Here it is noticed that \( \nu = 1.197 \ldots \) is determined around a loop, i.e. it is determined by \( 10.77/9 \), which is obtained by \( 4.77(2\text{nd column}) + 6 \), which \( 4.77 \) is calculated as \( 9.77 - 5 \), and, finally, we see that \( 9.77 \) is equal to \( 1.22 (= \frac{1}{8}) \times 8 \). (This loop is indicated by Greek letters \( \alpha, \beta, \gamma, \delta, \ldots \).) In such a case, we put \( \epsilon = 4 \), calculate \( \delta \) from \( \epsilon \) in the same way as \( 9.77 \) was calculated from \( 1.22 \), i.e.

\[
\delta = \epsilon \times 8 = 8\nu
\]

Similarly,

\[
\gamma = \delta - 5 = 8\nu - 5
\]

\[
\beta = \gamma + 6 = 8\nu + 1
\]

\[
\alpha = \beta \div 9 = \frac{8\nu}{9} + \frac{1}{9}
\]

Putting this \( \alpha \) equal to \( \epsilon \), we have

\[
\frac{4}{\nu} (= \alpha = \epsilon) = 1, \quad \beta = 9, \quad \gamma = 3, \quad \delta = 8
\]

Replacing the positions of \( \alpha, \beta, \gamma, \delta \) by these values, we restart the iteration as follows. 2)

1) These \( \infty \)'s should be written as \( M \) according to the rule explained in the preceding sections.
2) See §9.1.
\begin{align*}
\text{V5 \& C5} \\
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 5 & 5.77 & 6 & 1/22 & 1 & 29 \\
3 & 4.77 & 4.77 & 7 & \infty & 0/52 & 32 \\
7 & 8.77 & 8.77 & \infty & \infty & 36 \\
1 & 1 & 0 & 3 & 4.77 & 37 & 629 \\
\end{array}
\quad
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & 6 & 6 & 6 & 6/17 & 1/15 & 29 \\
5 & 6.875 & 7 & 7 & \infty & 0/52 & 32 \\
9 & \infty & \infty & 4 & \infty & \infty & 36 \\
10 & 11 & 11 & 11 & 5 & 5 & 38 \\
11 & 12.375 & 14 & 14 & 6/48 & 8 & 37 \\
1.22 & 1.375 & 0 & \infty & \infty & \infty & 581 \\
1.22 \times 18 = 22 \\
\end{array}
\end{align*}
\( \langle V_6 \ & C_6 \rangle \)

<p>| | | | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
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<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>30</td>
<td>(0, 17)</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>6</td>
<td>29</td>
<td>(0, 22)</td>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>6,875</td>
<td>7</td>
<td>2</td>
<td>(0, 52)</td>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
<td></td>
<td></td>
<td>36</td>
<td>(41, 0)</td>
<td>6</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>11</td>
<td>11</td>
<td>38</td>
<td>(70, 0)</td>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1,25</td>
<td>1,375</td>
<td>37</td>
<td>(0, 50)</td>
<td>9</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>160</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>563</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ 1.25 \times 160 = 200 \]

\( \langle V_7 \ & C_7 \rangle \)

<p>| | | | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
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<tbody>
<tr>
<td>0</td>
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<td>(0, 17)</td>
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<td>6</td>
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<tr>
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<td>6,875</td>
<td>7</td>
<td>32</td>
<td>(0, 52)</td>
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<td></td>
</tr>
<tr>
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<td></td>
<td></td>
<td></td>
<td>36</td>
<td>(41, 0)</td>
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</tr>
<tr>
<td>11</td>
<td>11</td>
<td>11</td>
<td>38</td>
<td>(50, 20)</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
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</tr>
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<td>48</td>
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<td></td>
<td>403</td>
<td></td>
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<td></td>
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</table>

\[ 1,375 \times 48 = 66 \]

- 76 -
Thus we have the optimum solution shown in (10.26). However, it is

<table>
<thead>
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<th></th>
<th>1</th>
<th>2</th>
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<th>4</th>
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<td>0</td>
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<tr>
<td>6</td>
<td>17</td>
<td>14</td>
<td>8</td>
<td>11</td>
</tr>
</tbody>
</table>

(10.26)

not the unique solution, since \( <C_{11} > \) may be performed in another way (for example, as shown in (10.27) together with the resulting optimum solution).

\[
< C_{11} >
\]

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
1 & 1 & 29 & 30 \\
2 & 22 & 0 & 0 \\
3 & 0 & 0 & 17 \\
4 & 0 & 41 & 0 \\
5 & 23 & 0 & 47 \\
6 & 0 & 14 & 3 \\
\end{array}
\]

(10.27)

The total cost corresponding to the optimum solutions is calculated directly from (10.25) and \( <V_1 & C_1 > \sim <V_{11} & C_1 > \):

\[
3162 + 0 + 22 + 240 + 48 + 22 + 200 + 66 + 220 + 20
\]
Next, let us consider "Problem B". The network structure for this problem may be as shown in Fig. 15, where only the part differing from Fig. 14 is drawn. As is readily seen, the optimum current configuration corresponding to

\[ E_0 - \sum \frac{h_j}{h_i} \times x_{i,j} = 355 \] is the same as before (the end of \( \langle V_7 \& C_7 \rangle \) for "Problem A"), and the total cost for that configuration is

\[ 3162 + (0 + 22 + 240 + 48 + 22 + 200 + 66) - (328 + 110 + 384 + 48 + 18 + 160 + 48) = 3760 - 1096 = 2664. \]

Then, the solution proceeds as follows.

---

1) The actual cost is 5275, 33 \( \text{M} \) being the term occurring due to the fictitious cost \( \text{M} \).
\[ V_8 \& C_8 \]

<table>
<thead>
<tr>
<th></th>
<th>2.33</th>
<th>6.66</th>
<th>17.75</th>
<th>17.75</th>
<th>19</th>
<th>30</th>
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</thead>
<tbody>
<tr>
<td>0</td>
<td>0/17</td>
<td>0</td>
<td>0</td>
<td>30</td>
<td></td>
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<td>9</td>
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<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ \varepsilon = \frac{3}{2}, \quad \varphi = \frac{3}{5} \]

\[ \varepsilon = \frac{5}{3} - 6, \quad \varphi = \frac{5}{3} + 5, \quad \chi = \frac{53}{8} + \frac{5}{8} = \varepsilon \]

\[ \therefore \frac{3}{3} = \frac{5}{3} = 1.66 \]

\[ 1.66 \times 132 = 220 \]
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Thus the optimum solution is given by (10.30) (which is unique).

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This solution coincides exactly with that obtained by Abadie in (14). The corresponding total cost is 5268, which is calculated from (10.29) and \( \langle V_8 & C_8 \rangle \sim \langle V_{11} & C_{11} \rangle \) as follows:

\[
3760 - 1096 n + 220 + 20 + 638 + 630 = 5268 - 1096 n.
\]

(10.31)

References

[1] Papers in Division A of the RAAG Memoirs, Vol. 10 (1955), and those in Divisions A and C of the RAAG Memoirs, Vol. 2 (1958). See also the papers in other Divisions (especially, Division F) and the Research Notes, Second and Third Series, circulated among the RAAG members. The papers containing the achievements of the RAAG activities after the publication of the RAAG Memoirs, Vol. 2, will be published in the refined form, in the coming volume, Volume 3.


(11) 藤口泰一, 電気計画法入門。日科技連ライブラリー 1, 東京, 1957.


The following Note, which includes several applications of the theory of general information networks and has close connexion with this Note from the methodological points of view, appeared after the manuscript of this Note has already been completed.

15 M. Iri, Several Applications of the Basic Theory of General Information Networks —— Connexion Propriétés of Graphs and Finite Deterministic Two-Person Games, RAAG Research Notes, Third Series, No. 40 (December 1960).
The following note, which introduces several applications of the theory of General Information Networks and the use of connection with this note from the methodology of these networks after the manuscripts of the note for emphasis on completing. 12, Chapter 3, several applications of the basic theory of General Information Networks — Connection Procedures of Groups and Multiple Person Interaction, Two Person Game, and Reservoir Notes, Third Series, No. 40 (December 1960)