An Anholonomic Statistical Approach to Space-Time Electromagnetism, Microscopic Physics and General Relativity Theory in Analogy with Incompatibility, Plasticity and Stress-Function

By

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應用幾何學研究協會
不適合、塑性、応力関数理論との類同による時空間。
電磁気、数値物理学、一般相対性理論の構造の
非 ローマノフ統計観測による研究

近藤一夫，甘利俊一

概要

非 リーマン塑性論の研究の中で [1], [2], 弾性-塑性
理誌で取扱われる不完全結晶の如き三次元物質連続体
と，一般相対性理論，統一度理論，量子理論に現われる
四次元時空間との間に，その理論構成からみて深い
類似があることがわかった。ここでは R A A G
Mémoires Vol. II の E 部門ですでにわれてきている
同系統の観測の幾何学 [3], [4]，の直接の発展を目的
とするのではなく，弾性-塑性理論と電磁気-相対性
理論との類同が単にみかけだけのものではなく，理論
構成のより深いところにその原因があり，これにより
全く独立に発展してきている両部門を統一的に研究
することが可能であることを強調し，その点に読者の
注目をひきたい。

ここで行われる試みは時空の計量と非 ローマノフ性より
生じた挙動の起原についての解釈を除いてはすでに
従来の統一度理論で試みられたことがあるかもしれないと
しかしそれにより三村の波動幾何学 [5] の観測に
よる基礎づけも行えると思われるし，また Einstein の
一般相対性理論に採用された理論構成 [6],[7],[8],[9]
を拡張することも出来るとと思われる。ここで使用する
幾何学は Cartan の幾何学 [10],[11],[12] のうちに含まれ
るものである。
An Anholonomic Statistical Approach to Space-Time Electromagnetism, Microscopic Physics and General Relativity Theory in Analogy with Incompatibility, Plasticity and Stress-Function

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INTRODUCTION

IN our pursuit of Non-Riemannian Plasticity Physics\textsuperscript{1)} we recognize a resemblance between the elasticity-theoretical and crystallographical anomalies of three-dimensional material continua, such as imperfect crystals, and those of four-dimensional space-time manifold which appear in General Relativity and Unified Field Theory as well as Quantum Physics. An investigation into cognate problems has been made in RAAG Memoirs Vol. II, Division E. Rather than attempt a direct extension of those papers on the Geometry of Observation, (3), (4), the authors will first draw the reader's attention to the elastic-plasticity and relativity-electromagnetism analogy, which they believe to be more than a formal one, and which could have earlier united the two fields of independent development.

Except some of the interpretations of the origin of space-time metric and the associated torsion and anholonomy affording also the observational basis of Mimura's Wave Geometry (5), the following theoretical structure is not entirely new and may have already been pointed out — most likely in some of the earlier publications on unified field theory. Being a little removed from the study of the literature in theoretical physics, which seems to be so extensive, the authors were unable to ascertain thoroughly which classical publications need to be mentioned in this connexion. As far as the authors are aware, the present exposition revives to some measure Einstein's attempt around 1928 ([6], [7], [8], [9]). It has been implicitly included in E. Cartan's geometry ([10], [11], [12], [13]). Incidentally, it does not seem to be directly connected with G. Vranceanu's formulae with anholonomic hypersurfaces, (14).

\textsuperscript{1)} See references [1] and [2].
1. Construction of an Anholonomic Space with Intrinsic Metric by Statistical Observation

Suppose that we have a manifold whose point is specified by a finite or countable number of variables

\[ \xi^\alpha \quad (\alpha = 1, \ldots, \nu \leq \infty). \]

The difference of arbitrary two neighbour points is then expressed by differentials

\[ d\xi^\alpha. \]

By statistical observation they can be summarized into

\[ \overline{w}^\alpha = w_\alpha^\alpha d\xi^\alpha \quad (\alpha = 1, \ldots, m, \ldots), \quad (1.1) \]

some of which may vanish

\[ 0 = w_\alpha^c d\xi^\alpha \quad (c = m+1, \ldots, N), \quad (1.1.1) \]

where \( w_\alpha^\alpha \)'s have the meaning of weighting function.

The statistical meaning of \( w_\alpha^\alpha \) may be explained as follows. Let \( \Lambda \) be the set of the possible (countable or uncountable) observations applied to the manifold in the neighbourhood of the point \( \xi^\alpha \). Let its elements be denoted by \( \lambda \)'s and suppose that the observation labelled with \( \lambda \) \((\in \Lambda)\) gives an weighting function \( w_\lambda^\alpha \) for \( d\xi^\alpha \). It defines formally a Pfaffian

\[ \overline{w}_\lambda^\alpha = w_\lambda^\alpha d\xi^\alpha. \]

But it is not a priori required that any specific element of \( \Lambda \) is preferred to another. An arbitrary element of \( \Lambda \) is generally as probable as another. Therefore, \( w_\alpha^\alpha \) or \( \overline{w}_\alpha^\alpha \) for a given \( d\xi^\alpha \) are probabilistic variables. It does not exclude that there are some degenerate observations giving \((1.1.1)\) for some \( c \).

The statistical meaning of \( \overline{w}^\alpha \) is that the probability density of finding the mean values \( \overline{w}^\alpha \) for the difference \( d\xi^\alpha \) has the distribution

\[ \overline{f} = f(\overline{w}^\alpha), \]

...
where \( f \) is an analytic function of \( \omega^a \)'s satisfying
\[
\int \Omega \, d\omega^1 \, d\omega^2 \ldots \, d\omega^m = 1 .
\] (1.2)

If the number of the parameters \( \xi^a \) is sufficiently large, the Central Limit Theorem is effective so that the distribution will become Gaussian; its variance matrix, as well as the inverse variance matrix, can be transformed into diagonal form by an appropriate local coordinate transformation, so that we have
\[
\Omega = \left( \frac{1}{\sqrt{2\pi h}} \right)^m e^{-\frac{f}{2}} .
\] (1.3)

where
\[
\bar{\xi} = -\frac{\delta_{ba} \omega^a \omega^b}{h^2} .
\] (1.4)

Here the \( \omega^a \) can be thought to be a deviation from the mean value. The variance associated with the point is sufficient to give a metric to the manifold. The larger the inverse variance, the longer the element \( d\xi^a \). A formal \( m \)-dimensional Pythagorean form
\[
ds^2 = \delta_{ba} \omega^a \omega^b ,
\] (1.5)

thus associated with the point \( \xi^a \), is a basic invariant of the observation. The Pfaffians \( \omega^a \)'s are, however, probabilistic functions which depend on the specific observation, i.e., on \( \lambda \). So is \( ds^2 \), or the tensor
\[
a_{\rho \kappa} = \delta_{ba} \omega^a \omega^b
\]

which gives
\[
ds^2 = a_{\rho \kappa} (\xi^{\rho} \xi^{\kappa}) d\xi^\rho \, d\xi^\kappa .
\] (1.6)

By adopting the Gaussian normal distribution, the \( \lambda \)'s have been eliminated. There will be no more mention of \( \lambda \) hereafter.

The tensor \( a_{\rho \kappa} \) is a function of \( \xi^a \). Hence, the picture so obtained is as anholonomic space (cf. [15]),
defined by (1.1) and (1.1.1) with an intrinsic invariant metric (1.5).

Although there seems to be no a priori restriction imposed upon the number \( m \), we shall study elsewhere that the Pfaffians are grouped into sets of 4 elements, each defining 4-dimensional Euclidean space and also that each such space has the characteristics of the spacetime of special relativity, i.e., it has the Minkowskies metric defined by the pseudo-Euclidean quadratic form with one time (-like) and 3 space (-like) coordinates:

\[
ds^2 = (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2 - (\omega^4)^2.
\]

In other words, we can restrict \( m \) to 4, and assume

\[
\delta_{\mu\nu} = \delta_{\mu\mu} = \delta_{\nu\nu} = 1, \quad \delta_{\mu\nu} = -1, \quad \delta_{\alpha\beta} = 0 \quad (\alpha \neq \beta),
\]

in (1.4).

Lemma 1.1. The fundamental observation constructs the anholonomic subspaces \( X^m_{\mu\nu\rho\sigma} \) defined by

\[
0 = \omega_{\rho} \delta^\mu \quad (c > 4)
\]

and having an intrinsic metric

\[
ds^2 = \omega_{\rho\sigma} \delta^\mu \delta^\nu,
\]

where \( \omega_{\rho\sigma} \) are functions of parameters \( \xi^\rho \).

Now that \( N-m \) \((m=4)\) of the \( N \)-differentials \( d\xi^\rho \) can be eliminated from (1.1) and (1.1.1) so that we have

\[
\omega^a = B^a_{\xi^\rho} dx^\rho \quad (a = 1; \ldots; 4) \quad (1.8)
\]

1) See Research Notes which will be published in the near future, cf. [3], [4], [16] and also [17], [18], [19] and especially [20].
where $x^\kappa$ are 4 of $\xi^\kappa$ and $B_\kappa^a$ are functions of $x^\kappa$ as well as of the rest of $\xi^\kappa$, which we shall hereafter denote by

$$\gamma^\kappa \quad (\kappa = 5, \ldots, N \leq \infty).$$

The fundamental metric form now becomes

$$ds^2 = \gamma_{\kappa\lambda} \, dx^\kappa \, dx^\lambda \quad (\kappa, \lambda = 1, \ldots, 4)$$

where

$$\gamma_{\kappa\lambda} = B_\kappa^a \, B_\lambda^b \delta_{ab}. \quad (1.9)$$

Lemma 1.2. The anholonomic space $\chi^{4_N}$ constructed by the fundamental observation has the intrinsic metric

$$ds^2 = \gamma_{\kappa\lambda} \, dx^\kappa \, dx^\lambda,$$

where the metric tensor $\gamma_{\kappa\lambda}$ is functions of 4 coordinates $x^\kappa (\kappa = 1, \ldots, 4)$ and a countable or uncountable number of implicit parameters $\gamma^\kappa$.

2. Lorentz and Gauge Invariance

The restriction (1.7) is not sufficient to determine $B_\kappa^a$ uniquely which can further be transformed to within arbitrary orthogonal transformations, $B_\kappa^{a'}$, such that

$$B_\kappa^a = B_{a'}^{a'} \cdot B_\kappa^b$$

where

$$\delta_{ba} B_{a'}^{a'} \cdot B_b^b = \delta_{b'a'} \quad (2.1)$$

so that

$$\gamma_{\kappa\lambda} = \delta_{b'a'} \cdot B_{\lambda}^{a'} \cdot B_{\kappa}^{a'} \quad (1.8.1)$$

The orthogonal transformation (2.1) means a Lorentz transformation. Hence
Lemma 2.1. The local anholonomy structure has Lorentz invariance.

In more general $B^a_b$ and $B^b_a$ of (2.1) can be replaced by any pair of operators such that

$$X^{a'}_a \cdot X^{b'}_b \overset{\text{def}}{=} \sum_{ba} X^a_a \cdot X^{b'}_b = \delta^{b'}_a \delta^a_b$$

where $X^{a'}_a$, $X^{b'}_b$ are the space-time components of the operator $X^{a'}_a$ and $X^{b'}_b$ respectively and $\overline{X^{b'}_b}$ is the complex conjugate of $X^{b'}_b$. One such pair is given by

$$X^{a'}_a = b^a_a \exp i\gamma, \quad \overline{X^{b'}_b} = b^b_b \exp -i\gamma,$$

where $b^a_a$ and $\gamma$ are real. Hence, our formulae are invariant under the transformation (2.3) which is a gauge transformation.

Lemma 2.2. The local anholonomy structure has Gauge invariance.

The more general possibilities of the operators $X$ and $\overline{X}$ will be obtained from the theory of representations of Lorentz and Gauge transformations, as will be studied elsewhere.

3. Least-Square Variational Criteria

The most probable configuration is given by the extremal distribution of the probability density. In fact from (1.3) by the variational principle we have

$$\delta \int \mathcal{Q} \frac{dX}{\xi} = 0,$$

where $\mathcal{Q}$ is normalized for the respective set of parameters $\gamma$, but the normalization is no longer preserved if these parameters are changed so that the variation in terms of $\gamma$ is meaningful. The dependence of $\mathcal{Q}$ on $\gamma$ will soon become apparent as the dependence of the metric tensor on them.

1) There is a certain reason that the logarithm of the normalization factor is assimilated by the square of the time coordinate in $ds^2$. 

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\[
d\chi = \ast d\varpi d\varpi' d\varpi' d\varpi^+ , \quad (3.2)
\]
and (3.1) gives
\[
\text{const.} \int_S \bar{\varpi}_a \delta \varpi^a d\chi = 0 . \quad (3.3)
\]

It should be noted that the variation in (3.3) is
affected by varying the parameters \( \gamma^f \). The fundamental
tensor \( g_{\lambda \kappa} \), as well as \( \varpi^a \), is varied owing to the changes
of these parameters but not to the changes of the coordinates. If the extremal condition is not satisfied, the
normalization may also fail.

The small variation effected on the metric tensor is
written
\[
\delta g_{\lambda \kappa} = \frac{\partial g_{\lambda \kappa}}{\partial \gamma^f} \delta \gamma^f = 2 \delta g_{\lambda \kappa} \varepsilon_{\lambda \kappa} \frac{\partial \varepsilon_{\lambda \kappa}}{\partial \gamma^f} \delta \gamma^f , \quad (3.4)
\]
or
\[
\delta (ds^2) = \delta g_{\lambda \kappa} dx^\lambda dx^\kappa = 2 \bar{\varpi}_a \delta \varpi^a . \quad (3.4.1)
\]
By substituting (3.4) or (3.4.1) in (3.3) we have
\[
\int (\Omega \bar{\varpi}_a \frac{\partial \varpi^a}{\partial \gamma^f}) \delta \gamma^f d\chi = 0 .
\]
Now that \( \Omega \delta \gamma^f \) are arbitrary, we have
\[
\bar{\varpi}_a \frac{\partial \varpi^a}{\partial \gamma^f} = 0 . \quad (3.5)
\]
Hence, we always have
\[
\left( \frac{1}{2} \delta (\bar{\varpi}_a \varpi^a) = \right) \bar{\varpi}_a \delta \varpi^a = 0 . \quad (3.5.1)
\]
Equation (3.5) is nothing but the principle of least
squares for the observation of \( d\xi^a \).

Now that \( \bar{\varpi}_a \) is a vector in the local space-time,
let us write
\[
dA \quad \text{for} \quad \bar{\varpi}_a
\]
and
\[
\psi \quad \text{for} \quad \delta \varpi^a .
\]

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Then (3.5.1) becomes
\[ ds \cdot \psi = 0 \]  
(3.5.2)

This is also subject to Lemmas 2.1 and 2.2, i.e. the equation (3.4.1) must satisfy Lorentz and Gauge invariance.

Lemma 3.1. The parametric structure of the observation is regulated by
\[ ds \cdot \psi = 0 \]
satisfying the general Lorentz and Gauge invariance.

It is evident that the restriction (3.5.2) under the Lorentz and Gauge invariance agrees with the fundamental postulate of Mimura's Wave Geometry, [5], (cf. also [20]).

THEOREM 1. The postulate of Wave Geometry is identical with the principle of least squares of the observation of the differences of the parameters of the basic manifold.

4. Metric Connexion

In the preceding section, the parameters \( \gamma' \), which have initially been unfixed, have become more restricted by assuming the extremal probability distribution. Hence the metric has approached to the simple Riemannian case. They can be fixed in \( g_{\lambda\kappa} \) at least, if not in \( \psi' \) and \( ds' \).

In other words, we have arrived at

Lemma 4.1. The fundamental observation can be refined to a four-dimensional metric space having an invariant metric defined by
\[ ds^2 = g_{\kappa\lambda} \, dx^\kappa \, dx^\lambda \quad (\kappa, \lambda = 1, \ldots, 4) \]

where
\[ g_{\kappa\lambda} = \delta_{ab} \, B_{\kappa}^a \, B_{\lambda}^b \]
as well as an implicit local structure
\[ ds \cdot \psi = 0 \]
satisfying General Lorentz and Gauge invariance.
It was from this point that Mimura started on his search of the wave-geometrical structure.

By assuming the parameters $\eta'$ as fixed in $\varepsilon_{\mu}^{\kappa}$, one can define an intrinsic parallelism of the space which preserves the metric invariance by

$$
\nabla_{\mu} \varepsilon_{\lambda \kappa} \overset{\text{def.}}{=} \partial_{\mu} \varepsilon_{\lambda \kappa} - \Gamma_{\mu \lambda}^{\nu} \varepsilon_{\nu \kappa} - \Gamma_{\mu \kappa}^{\nu} \varepsilon_{\lambda \nu} = 0, \quad (4.1)
$$

where $\partial_{\mu} = \frac{2}{\varepsilon_{\mu}^{\kappa}}$, $\Gamma_{\mu \lambda}^{\nu}$ are the coefficients of the linear connexion.\textsuperscript{1)} Solving (4.1) with respect to $\Gamma_{\mu \lambda}^{\nu}$ one gets

$$
\Gamma_{\mu \lambda}^{\nu} = \frac{1}{2} \varepsilon_{\lambda \nu} (\partial_{\mu} \varepsilon_{\nu \kappa} + \partial_{\nu} \varepsilon_{\mu \kappa} - \partial_{\kappa} \varepsilon_{\mu \nu} ) \quad (4.2)
$$

where

$$
\frac{1}{2} \varepsilon_{\lambda \nu} = \frac{1}{2} \varepsilon_{\lambda \nu} \quad (4.3)
$$

and $S_{\mu \lambda}^{\nu \kappa}$ is a tensor which is antisymmetric with respect to the lower indices and is called the torsion tensor. So far the observation has not restricted the torsion tensor any further, which means

Lemma 4.2. Observation admits an arbitrary torsion.

On restricting the observation by something more than the Least Square Criterion, torsional characteristics can further be restricted. We first mention the following extremal cases:

1° Levi-Civita Parallelism:

$$
S_{\mu \lambda}^{\nu \kappa} = 0;
$$

2° Distant parallelism:

$$
R_{\mu \lambda}^{\nu \kappa} = 0,
$$

where

$$
R_{\mu \lambda}^{\nu \kappa} = 2 \left( \partial_{\mu} \Gamma_{\nu \lambda}^{\kappa} + \Gamma_{\mu \nu}^{\gamma} \Gamma_{\gamma \lambda}^{\kappa} \right) \quad (4.4)
$$

\textsuperscript{1)} See, e.g., [22], [23].
$R^{\kappa\lambda\mu\nu}$ is a tensor called the Riemann-Christoffel curvature tensor, as is well known. $S^{\kappa\lambda\mu\nu}$ is equal to the anti-symmetric part of the parameters of connexion:

$$S^{\kappa\lambda\mu\nu} = \varepsilon^{\kappa\lambda\mu\nu} \Gamma_{\nu\lambda\mu},$$

as far as the reference frame is holonomic.

The roles of the two tensors are well known. They give respectively the discrepancies of location and of direction after describing a small circuit by parallel displacement. If the circuit encircles the area expressed by the bivector

$$f^{\kappa\lambda} = f \left[ \gamma^{\kappa\lambda} \right],$$

then the discrepancy of location is given by

$$\Delta \omega^{\kappa} = f^{\kappa\lambda} S^{\kappa\lambda\mu\nu}$$

and that of direction by

$$\Delta \omega_{\kappa} = f^{\nu\kappa} R^{\kappa\nu\mu\lambda}$$

so that we have the change

$$\Delta U^{\kappa} = f^{\gamma\kappa} R^{\kappa\gamma\mu\lambda} U^{\lambda}$$

for an arbitrary vector $U^{\lambda}$, by displacing it pseudo-parallelly along the circuit.

As is well known, the Riemann-Christoffel curvature tensor for the Levi-Civita Parallelism, which we shall denote by

$$K^{\kappa\lambda\mu\nu} = 2 \left( \partial_{\nu} f^{\rho\lambda} \right)_{\kappa} + \frac{1}{2} \left( \Gamma^{\kappa}_{\rho\lambda} \right)_{\nu} f^{\rho\mu}$$

in the following, does not usually vanish. Such a space that $K^{\kappa\lambda\mu\nu} \neq 0$ and $S^{\kappa\lambda\mu\nu} = 0$, is called Riemannian.

On going to an arbitrary anholonomic reference frame, say ($\kappa'$):

$$\omega^{\kappa'} = C^{\kappa'}_{\kappa} \, dx^{\kappa}, \quad dx^{\kappa} = C^{\kappa}_{\kappa'} \, \omega^{\kappa'} (\kappa' = 1, 2, 3, 4)$$

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we have
\[ \bar{\Gamma}_{\mu' \lambda'} = \{ \mu' \lambda' \} - \Omega_{\mu' \lambda'} + S_{\mu' \lambda'} - 2 S_{\epsilon \lambda' \lambda}, \]
so that
\[ S_{\mu' \lambda'} = C_{\mu' \lambda'}, \]
where
\[ \Omega_{\mu' \lambda'} = C_{\mu' \lambda'}, \]
and \( \{ \mu' \lambda' \} \) has apparently the same form as (4.3) provided \( \varpi_{\mu'} \) etc. are replaced by
\[ \varpi_{\mu'} = C_{\mu' \varpi}, \]
etc.

The corresponding modification of the expression of the Riemann-Christoffel curvature tensor, defined initially by (4.4), becomes
\[ R^\kappa_{\mu' \lambda'} = 2 (C_{\mu' \lambda'} + \Sigma_{\mu' \lambda'} + \Omega_{\mu' \lambda'}) + \Sigma_{\kappa \mu' \lambda'}. \]

It cannot happen that \( R_{\mu' \lambda'} \) vanishes identically so far as \( R_{\mu' \lambda'} \) is not a zero tensor, however, such can happen if the space has distant parallelism. If it is possible to select a particular anholonomic frame such that
\[ \omega_{\mu' \lambda'} = A_{\mu' \lambda'}, \]
we have
\[ \bar{\Gamma}_{\mu' \lambda'} = \{ \mu' \lambda' \} = \frac{1}{2} \delta_{\mu' \lambda'} (\varpi_{\mu'} \varpi_{\lambda'} - \varpi_{\mu'} \varpi_{\lambda'}) \]
By further restricting the anholonomic frame so that (k'') agrees with the (a) defined above - in (1.1) and (1.1.1) - we have
\[ \bar{\Gamma}^{a}_{cb} = \left\{ \frac{a}{db} \right\} = 0, \]

as has been pointed out in Research Note No. 34, [24].

By virtue of (4.4.1), we shall then have

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\[
R_{dcb}^a = 0 ,
\]
which means distant parallelism. The corresponding torsion tensor is also a specific one and is defined by
\[
Z_{cb}^a = \Pi_{cb}^a \overset{\text{def}}{=} B_d^a \frac{\partial}{\partial x^d} .
\] (4.6)

These considerations may be summed up into:

Lemma 4.3. If the space of observation defined by Lemma 4.1 is further restricted by specifying the torsion tensor to
\[
Z_{cb}^a \overset{\text{def}}{=} \Pi_{cb}^a ,
\]
it obtains distant parallelism.

Definition. We shall call the procedures of so specifying the torsion tensor in connexion with an anholonomic reference frame "tearing". The special tearing of Lemma 4.3 will be called "perfect tearing".1)

To sum up, we have

Theorem 2. The observational 4-manifold is Riemannian if its implicit anholonomic structure is neglected. By taking account of the latter, it can be modified to a space of distant parallelism with the same metric. It has further implicit internal structure pointed out by Wave Geometry.

Problems remain for the possibility of (4.6) for an arbitrary given torsion tensor. Is it always possible to find such an anholonomic frame that its anholonomic object equals the torsion tensor (not invariably of course but in the particular reference frame so concerned)? The possibility depends on the number of dimensions of the space. If \( k'' \) is fixed we obtain \( n(n-1)/2 \) equations of (4.6):

1) cf. (24) where the perfect tearing of plastic manifold is defined.
$$\partial_{\gamma} A^\gamma = S^k_{\gamma k} A^k = 0,$$  \hspace{1cm} (4.6.1)

one for each \( \gamma \). The number does not necessarily agree with \( n \) (see Table I). Hence, with the exception of the three-dimensional case, an arbitrary torsion tensor are not generally be expected to be torn by means of an anholonomic frame. In the four-dimensional space-time we have 6 equations of (4.6.1) for four unknown \( A^\kappa \) (where \( \kappa \) is fixed) so that the torsion tensor obtained by tearing is not expected to be a general one. In three-dimensions the number of degrees of freedom of \( S^k_{\gamma k} \) is the same as that of \( \Omega^\gamma \). Hence in plasticity problems, in which the dislocation density is compared with the latter can be defined by tearing in concordance with the aboriginal concept of the dislocation, provided that such further conditions for the existence of the solution of (4.6.1), as might be imposed, do not contradict it.

In the following we shall restrict ourselves to the torsion tensors obtained by tearing although they need not necessarily obtained by perfect tearing. We have the general and rigorous relation

$$\hat{R}^\xi_{\nu \mu \lambda} = 2 \hat{V}^\xi_{\nu \mu \lambda} - 4 \mathcal{S}^\xi_{\nu \mu \lambda},$$  \hspace{1cm} (4.9)

which holds identically for any space with the linear connexion (4.2). It can be put in a compact form in terms of Cartan's operator

$$\hat{R}^\xi_{\nu \mu \lambda} = 2 \hat{V}^\xi_{\nu \mu \lambda},$$ \hspace{1cm} (4.9.1)

where \( \hat{V} \) means that the alternating part \( S^\xi_{\nu \mu \lambda} \) of \( \Gamma^\xi_{\nu \lambda} \) does not apply to the lower indices. In more specific, the anholonomic structure which may lie in the background of \( S^\xi_{\nu \mu \lambda} \) is not concerned with the differentiation in the direction of \( \mu \), \( \nu \), \( \lambda \) while the torsional and curvature

1) See (11), p.211 and also (23), p.150.
discrepancies characteristics represented by the upper index \( \kappa \) are not insensitive of the defining structure.

Here we should remember that the torsion tensor field satisfying (4.9.1) is not unique. If it is obtained by perfect tearing, it can vary within the limit of the Lorentz and Gauge invariance of Lemmas 2.1 and 2.2.

**Lemma 4.4.** The torsion tensor is determinate only to within an arbitrary distant parallelism term \( S_{\mu \lambda}^{\kappa} \) such that

\[
0 = \hat{\nabla}_\nu S_{\mu \lambda}^{\nu \kappa} .
\]

5. Electromagnetism

By restricting the torsion tensor in the manner mentioned in §4, we have an alternating quantities defined by

\[
X_{\mu \lambda} \overset{\text{def}}{=} S_{\mu \lambda}^{\nu \kappa} A_{\kappa}^{\nu} = \partial_{\mu} A_{\lambda}^{\nu} . \tag{5.1}
\]

where \( k'' \) is fixed. \( A_{\kappa}^{\nu} \) with \( k'' \) fixed is evidently a covariant vector in regard to the lower index while the upper index refers to a fixed frame and hence it is suppressed in the notation of \( X_{\mu \lambda} \). A divergence of \( X_{\mu \lambda} \) defined by \( \hat{\nabla}_\nu X_{\mu \lambda} \) is identically annulled.

\[
\hat{\nabla}_\nu X_{\mu \lambda} = \partial_{\nu} X_{\mu \lambda} = 0 . \tag{5.2}
\]

while that defined by \( g^{\nu \rho}(\hat{\nabla}_\nu X_{\rho \lambda}) \) does not necessarily do so\(^1\): Let it be denoted by

\[
s_\lambda = g^{\nu \rho}(\hat{\nabla}_\nu X_{\rho \lambda}) . \tag{5.3}
\]

In deriving (5.2) it has been taken account of that the operator \( \hat{\nabla}_\nu \) is a special case of the more general \( \hat{\nabla}_\nu \) defined by

\[
\hat{\nabla}_\nu P_{\lambda_1 \cdots \lambda_p} \overset{\text{def}}{=} \hat{\nabla}_\nu P_{\lambda_1 \cdots \lambda_p} + \partial_{\nu} P_{\lambda_1 \cdots \lambda_p} + \sum_{i=1}^{p} S_{\nu \lambda_i}^{\kappa} X_{\kappa \lambda_1 \cdots \lambda_i \cdots \lambda_p} .
\]

\(^1\) Here, we define

\[
\hat{\nabla}_\nu X_{\lambda_1 \cdots \lambda_p} \overset{\text{def}}{=} \hat{\nabla}_\nu X_{\lambda_1 \cdots \lambda_p} + \sum_{i=1}^{p} S_{\nu \lambda_i}^{\kappa} X_{\kappa \lambda_1 \cdots \lambda_i \cdots \lambda_p} .
\]
reduces to
\[ \dot{V}_{\lambda_1 \cdots \lambda_f} = \omega_{\lambda_1 \lambda_2} P_{\lambda_2} \cdots \lambda_f \]
as is easily proved. Noting that \( X_{\mu \lambda} \) is a bivector, we therefore have
\[ \dot{V}_{\lambda_1 \cdots \lambda_f} = \omega_{\lambda_1 \lambda_2} X_{\mu \lambda_2} \]
The full analytical construction of \( s_{\lambda} \) is as follows.
We have
\[ \dot{V}_{\lambda} X_{\mu \lambda} = (\dot{V}_{\lambda} S_{\mu \lambda}) \Lambda_{\lambda}^{\kappa} + (\dot{V}_{\lambda} \Lambda_{\lambda}^{\kappa}) S_{\mu \lambda} \Lambda_{\lambda}^{\kappa} \]
Hence
\[ s_{\lambda} = (\dot{V}_{\lambda} S_{\mu \lambda}) \Lambda_{\lambda}^{\kappa} + S_{\alpha \beta} \Lambda_{\lambda}^{\kappa} \]
On the hand, \( \dot{V}_{\lambda_1 \cdots \lambda_f} \) is the same as \( \omega_{\lambda_1 \lambda_2} X_{\mu \lambda_2} \) and identically vanishes. Hence
\[ \dot{V}_{\lambda_1 \cdots \lambda_f} S_{\mu \lambda} \Lambda_{\lambda}^{\kappa} + (\dot{V}_{\lambda} \Lambda_{\lambda}^{\kappa}) S_{\mu \lambda} \Lambda_{\lambda}^{\kappa} = 0 \]
or, by taking account of (4.9.1),
\[ \frac{1}{2} R_{\mu \nu \lambda} \Lambda_{\lambda}^{\kappa} + (\dot{V}_{\lambda} \Lambda_{\lambda}^{\kappa}) S_{\mu \lambda} = 0 \]

1) We have
\[ \dot{V}_{\lambda_1 \cdots \lambda_f} = \dot{V}_{\lambda_1 \cdots \lambda_f} P_{\lambda_1} \cdots \lambda_f \]
But
\[ \dot{V}_{\lambda_1 \cdots \lambda_f} = \dot{V}_{\lambda_1 \cdots \lambda_f} P_{\lambda_1} \cdots \lambda_f - \Gamma_{\lambda_1 \lambda_2} \cdots \lambda_f - \Gamma_{\lambda_1 \lambda_2} \cdots \lambda_f \]
\[ = \omega_{\lambda_1 \lambda_2} P_{\lambda_1} \cdots \lambda_f + \frac{1}{2} \Gamma_{\lambda_1 \lambda_2} \cdots \lambda_f \]
Therefore,
\[ \dot{V}_{\lambda_1 \cdots \lambda_f} = \omega_{\lambda_1 \lambda_2} P_{\lambda_1} \cdots \lambda_f - \sum_{\lambda_1 \cdots \lambda_f} S_{\mu \lambda} \omega_{\lambda_1 \lambda_2} \cdots \lambda_f \]
Inserting (iii) into (i), we obtain
\[ \dot{V}_{\lambda_1 \cdots \lambda_f} = \omega_{\lambda_1 \lambda_2} P_{\lambda_1} \cdots \lambda_f, \quad \text{q.e.d.} \]

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This restriction is due to the selection of the specific torsion tensor. It need not be the same as distant parallelism which requires
\[ \mathcal{L}_k S^{\mu \nu} = -\left( \nabla_{(\nu} A_{\mu)}^{\alpha^\prime} \right) S^\alpha_{\mu \nu} = 0. \]

The set of relations
\[ X^\alpha_{\nu \lambda} = \partial_{\nu} A^\alpha_{\lambda} \quad (k'' \text{ fixed}), \]
\[ \partial_{\lambda} X^\alpha_{\nu \lambda} = 0, \]
\[ g^{\mu \nu} \partial_{\nu} X^\alpha_{\mu} = s_{\alpha}, \]
\[ (5.5) \]
is described in terms of the covariant differentiation with the symmetric parameters of connexion \( \Gamma^{\alpha}_{\beta \gamma} \) and has invariant meaning as far as the reference frame is holonomic. Notice that \( \partial_{\nu} \) and \( \partial_{\lambda} \) in the first two equations are the same as \( \hat{\nabla}_{\nu} \) and \( \hat{\nabla}_{\lambda} \). The structure is not preserved, however, if an anholonomic reference frame is used.

An important significance of (5.5) should be remarked that it reduces to the same structure as the Maxwell equations of electromagnetic fields when the disturbances, i.e., the Riemannian deviation of the metric tensor from the Euclidean restriction, namely
\[ \xi^{\lambda \kappa} = g^{\lambda \kappa} - \xi^{\lambda \kappa} \]
and the torsion tensor or \( X_{\mu \lambda} \) are sufficiently small. Since the latter disturbance has been defined in connexion with the imperfect tearing (4.6), its order of magnitude in comparison with that of \( \xi^{\lambda \kappa} \) could be made only by tearing the latter perfectly. Assuming small disturbances is assuming smallness of the perfect and imperfect tearings that come into the sphere of our observation.

In such a weak field, (5.5) is simplified approximately to
\[ X^\alpha_{\nu \lambda} = \partial_{\nu} A^\alpha_{\lambda}, \]
\[ \partial_{\lambda} X^\alpha_{\nu \lambda} = 0, \]
\[ \delta^r \partial_{\nu} X^\alpha_{\mu} = s_{\alpha}, \]
\[ (5.5.1) \]
It can be put in a more ordinary language in terms of Gibbsian vector notations as follows:

\[ \mathbf{H} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\nabla \phi - \partial_\mu A_\mu, \quad (5.6) \]

\[ \nabla \cdot \mathbf{H} = 0, \quad \nabla \times \mathbf{E} + \partial_\mu \mathbf{H} = 0, \quad (5.7) \]

and

\[ \nabla \times \mathbf{H} - \partial_\mu \mathbf{E} = \mathbf{s}, \quad \nabla \cdot \mathbf{E} = p, \quad (5.8) \]

where

\[ A_\mu = iA_1 + jA_2 + kA_3, \quad \Phi = A_4, \quad (5.9) \]

are for the \( A_\mu \) and \( (E, H) \) for \( X_{\mu \lambda} \).

Equations (5.6–9) having entirely the same structure as the Maxwell equations, the three-dimensional vectors and scalars appearing in them can be compared with the physical quantities in the theory of electromagnetism. \( \mathbf{E} \) and \( \mathbf{H} \) are thought to be the electric and magnetic field vectors, \( \rho \) the charge density and \( \mathbf{s} \) the current density, the velocity of light being taken as unity. \( \phi \) summarises the scalar and vector potential, \( \Phi \) and \( \mathbf{A} \). The four vector which is decomposed to \( \rho \) and \( \mathbf{s} \) can be compared with the four-current in special relativity. The two-index object \( X_{\mu \lambda} \) is equivalent to the alternating space-time tensor for the electromagnetic field intensity and derived from the four-vector \( A_\mu \).

If the introduction of the assumption of weak fields seems unsatisfactory, one may try an alternative to arrive at the same conclusion, essentially.

If the initial definition of the torsion tensor is referred to a perfect tearing, we have

\[ S_{\mu \nu}^{\cdot k} = Z_{\mu \nu}^{\cdot k} \]

and, in terms of the local Lorentz frame (a) which is anholonomic, \( \nabla_a \) is reduced to \( \mathcal{E}_a = F_k^a \partial /\partial x^k \). If we adopt \( g^{\nu \lambda} \mathbf{\nabla}_\nu X_{\lambda \rho} \) as the second divergence in place of \( g^{\nu \lambda} \mathbf{\nabla}_\nu X_{\lambda \rho} \), we get
\[ \delta^\alpha \varepsilon X_{\alpha \beta} = Z_\alpha, \quad (5.8.1) \]

in place of the third equation of (5.5), where \( Z \) is the counterpart of \( s_\alpha \) in the revised definition. (5.8.1) has evidently a similar form to (5.6) while the correspondingly revised first and second equation of (5.5) are

\[ X_{\alpha \beta} = \Pi_{\alpha \beta} \quad (i.e. \quad X_{\mu \nu} = \varepsilon_{\mu \nu \beta} B_\beta) \quad (5.6.1) \]

and

\[ \varepsilon_{\alpha \nu \lambda} X_{\alpha \nu \lambda} = 0, \quad (5.7.1) \]

where the latter equation has been obtained from

\[ \varepsilon_{\alpha \nu \lambda} X_{\alpha \nu \lambda} = \varepsilon_{\alpha \nu \lambda} X_{\alpha \nu \lambda} + \Pi^{\nu \lambda}_{\alpha \nu \lambda} X_{\nu \mu} - \Pi^{\nu \mu}_{\alpha \nu \lambda} X_{\nu \mu} = \varepsilon_{\alpha \nu \lambda} \left( \Pi^{\nu \lambda}_{\alpha \nu \lambda} X_{\nu \mu} + \Pi^{\nu \mu}_{\alpha \nu \lambda} X_{\nu \mu} \right) = \varepsilon_{\alpha \nu \lambda} X_{\alpha \nu \lambda}. \]

We note that if \( \varepsilon_{\alpha \nu \lambda} \) were to be identified with the partial differentiation operator \( \partial / \partial x^\alpha \), the revised definition would agree with the classical Maxwellian scheme. However, the frame (c) being anholonomic, such is not the case.

The first invariant definition which is based on \( \hat{\nabla}_k \)
has the advantage that \( \Gamma_{\mu \nu \lambda} = S_{\mu \nu \lambda} \), which is \( 0(X_{\mu \nu}) \), multiplyed by \( 0(X_{\mu \nu}) \) is automatically eliminated in so doing and hence the resulting equations are mostly linear in \( X_{\mu \nu} \) with the exception of \( g^{\mu \nu} \varepsilon_{\mu \nu \lambda} X_{\mu \nu} \). Replacing \( g^{\mu \nu} \) by \( \delta^{\mu \nu} \) and \( \varepsilon_{\mu \nu \lambda} \) by \( \varepsilon_{\mu \nu \lambda} \) is also equivalent to ignoring \( 0(X_{\mu \nu}) \) in comparison with \( 0(\delta_{\mu \nu}) \). Similarly we need to replace \( B_{\alpha} \) by \( s_{\alpha} \) ignoring \( B_{\alpha} s_{\alpha} = 0(X_{\alpha \beta}) \) in comparison with \( 0(s_{\alpha}) \) in order to reduce the second invariant definition to the Maxwellian form. The reducibility is therefore the weakness of the anholonomy in either case.

Summing up,

**Lemma 5.1.** Electromagnetism originates by tearing of the Riemannian space-time and the tearing need not be perfect;

and also
Lemma 5.2. It is possible to consider a number of generalizations of the classical equations of Maxwell for strong disturbances in invariant terminology. One is (5.5) in terms of Cartan and Schouten's operation $\mathcal{V}$ without reference to distant parallelism. Another is (5.6.1) - (5.8.1) in terms of $\mathcal{V}_e$ with reference to distant parallelism. Both are reduced to (5.5.1) or (5.6) - (5.9) for weak disturbances.

The second definition might be preferred to the first for formality's sake. However, we need not distinguish between them essentially, the origin of the torsion tensor by tearing being sufficient to cover all while the classical Maxwellian formulae cover only the local characteristics.

Once the weakness of the disturbances is assumed, the difference of perfect tearing is not very meaningful. In such a situation the Lorentz and Gauge invariance condition of §2 applies to the final (5.6) - (5.9) as follows.

Consider the scalar product

$$A^2 = A_\nu A^\nu = g_{\mu\nu} A^\mu A^\nu.$$

With reference to an arbitrary local Lorentz frame (a), such as studied in §2, it is given by

$$A^2 = \delta_{ab} A^a A^b.$$

Since such a frame is indeterminate to within Gauge and Lorentz Transformations,

$$A^a e^{i\varphi}$$

and

$$A^b e^{-i\varphi}$$

can be substituted for $A^a$ and $A^b$, as were $\omega^a e^{i\varphi}$ and $\omega^b e^{-i\varphi}$ for $\omega^a$ and $\omega^b$ in §2.

From the Lorentz invariance postulate it follows that

$$\mathcal{E}^a A_a = \nabla \cdot A + j_4 \mathcal{F} = \sigma$$

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must be a space-time invariant. The Gauge invariance requires that it must be equal to
\[ \delta^a (A_a C^b) = e^b \Phi (v \cdot A + \partial_t \Phi - i(A \cdot \nabla \phi + \phi \partial_t \phi)) = \sigma + i \zeta \]
in spite of arbitrary \( \phi \). Hence
\[ \sigma = \zeta = 0 \]
i.e. \( \phi \) is a constant and
\[ v \cdot A + \partial_t \Phi = 0 \tag{5.10} \]
or
\[ \partial_a A^a = 0 \tag{5.10.1} \]
Putting
\[ A_a = A'_a - \partial_a \chi \tag{5.11} \]
one can choose \( \chi \) arbitrarily as far as it satisfies
\[ \Box \chi = 0 \tag{5.12} \]
where
\[ \Box = \delta^{ab} \partial_a \partial_b = \nabla^2 - \partial_t \partial_t \]
The equation (5.10) or (5.10.1) is called the Lorentz condition and the invariance under the substitution (5.11)
or
\[ A'_a = A_a + \text{grad} \chi, \quad \Phi'_a = \Phi + \partial_t \chi \tag{5.11.1} \]
with (5.12) is the so-called Gauge transformation in the theory of electromagnetism.

From (5.6) and (5.8) we have
\[ \Box A = S - \text{grad} (v \cdot A + \partial_t \Phi) \]
where
\[ \Box = \nabla^2 - \partial_t \partial_t \]
\[ - 21 - \]
Similarly from (5.7) and (5.8) we have

$$\Phi = p + \partial_t (\nabla \cdot A) + \partial_t \Phi.$$  

Hence, by virtue of (5.10),

$$\square A = -S, \quad \square \Phi = p. \quad (5.13)$$

**THEOREM 3.** All of the electromagnetism follows from observation as the local characteristics of the weak field of the torsion tensor. Notice "weak" and "local" have almost the same meaning.

**Corollary.** Local Maxwell-Lorentz electromagnetic fields are connected anholonomically from point to point in space-time having Riemannian metric.

We must also notice that index $\kappa$ has been suppressed in the expressions of electromagnetic quantities. It means that an arbitrary vector may be multiplied in transvections to them. We may be justified in comparing it with a dimension of electromagnetism so that all electromagnetic quantities, such as $X_{\mu \lambda}$, as well as $s_{\kappa}$ could be proportional to a universal constant which we argue to be the electron charge $e$.

6. **Macroscopic Observation Leading to the Variational Criterion of General Relativity Theory**

6.1. **The Einstein Tensor.** Let $\nabla$ means the covariant differentiation with respect to the Levi-Civita parallelism. The covariant quantities in terms of $\nabla$ will be referred to in the following as "$\nabla$-covariant".

Put

$$R_{\mu \nu \lambda \kappa} = K_{\mu \nu \lambda \kappa} + L_{\mu \nu \lambda \kappa}. \quad (6.1)$$

Then

$$L_{\mu \nu \lambda \kappa} = 2 \left( \nabla_{\mu} \nabla_{\nu} \lambda \kappa + T_{\nu} \nabla_{\lambda \mu} \nu \kappa \right). \quad (6.2)$$

$$T_{\mu \nu \lambda \kappa} \quad \text{def} \quad \frac{\gamma_{\mu \lambda}}{\gamma_{\nu \kappa}} - \{\frac{\gamma_{\mu \kappa}}{\gamma_{\nu \lambda}}\} = S_{\mu \nu \lambda \kappa} - 2 S_{\nu \kappa \lambda \mu}. \quad (6.3)$$
is a tensor and distant parallelism need not be assumed. The respective Ricci tensors and scalar curvatures are defined by

\[ K_{\mu\lambda} = g^{\kappa\nu} K_{\nu\lambda\kappa\lambda} = g^{\kappa\nu} K_{\mu\nu\kappa\lambda} \]  \hspace{1cm} (6.3)

\[ R_{\mu\lambda} = g^{\kappa\nu} R_{\nu\mu\lambda\kappa} \]  \hspace{1cm} (6.3.1)

\[ K = g^{\lambda\nu} K_{\mu\lambda} \]  \hspace{1cm} (6.4)

\[ R = g^{\lambda\nu} R_{\mu\lambda} \]  \hspace{1cm} (6.4.1)

where

\[ R_{\mu\lambda} - K_{\mu\lambda} = L_{\mu\lambda} \]  \hspace{1cm} (6.5)

and

\[ R - K = L \]  \hspace{1cm} (6.6)

In going from (6.1) to (6.5) and (6.6) the microscopic structures of the manifold become less and less visible until at least in the relation

\[ \int K \sqrt{-g} \, d\chi = \int (R - L) \sqrt{-g} \, d\chi \]

where

\[ \sqrt{-g} = |g_{\lambda\kappa}| \]

\[ \sqrt{-g} \, d\chi = \sqrt{-g} \, dx^1 \, dx^2 \, dx^3 \, dx^4 \]

is the invariant four-dimensional volume element; they are entirely hidden. It should also be noted that the structure of \( L_{\nu\lambda\kappa\lambda} \) or \( L_{\nu\mu\lambda\kappa} \) are more microscopic than that of \( K_{\nu\lambda\kappa\lambda} \) or \( K_{\mu\lambda} \), in that the former depends on \( A_\kappa \) or \( S_\mu \) while the latter depends solely on the components of the metric tensor and their derivatives. Therefore, as far as the most macroscopic observation is concerned, we need to know only the macroscopic summary of \( R_{\nu\lambda\kappa\lambda} \), \( R_{\mu\lambda} \), \( L_{\nu\lambda\kappa\lambda} \), \( L_{\mu\lambda} \) etc. corresponding to \( K_{\nu\lambda\kappa\lambda} \), \( K_{\mu\lambda} \) etc., retaining the metric tensor as the sole unknown tensor. Hence,

\[ \frac{m(m+1)}{2} (m = 4) \]

\[ \Rightarrow 23 \]
linearly equations are necessary and sufficient, so that a set of equations in terms of \( K_{\cdot\cdot} \), which is symmetric and hence has 10 linearly independent component, in place of \( K_{\cdot\cdot\cdot} \), which has 24 linearly independent components, will suffice. Among such possible sets, the simplest possible case has been resorted to by Einstein who assumed that the tensor

\[ G_{\cdot\cdot} = K_{\cdot\cdot} - \frac{1}{2} K g_{\cdot\cdot} \]  

or the tensor density

\[ \sqrt{-g} \ G = \sqrt{-g} (K_{\cdot\cdot} - \frac{1}{2} K g_{\cdot\cdot}) \]  

(6.7)

is to be given a priori by some physical reason. Equation (6.7) is called Einstein's field equation and the tensor \( G_{\cdot\cdot} \) the material-energy tensor. It was because of the non-divergence of \( G \):

\[ \nabla_{\cdot} G^\cdot = \nabla_{\cdot} (K_{\cdot\cdot} - \frac{1}{2} K g_{\cdot\cdot}) = 0 \]  

(6.8)

that he sought in the equation a particular physical significance. As is well known, it can be derived as the Euler equation from the variational criterion

\[ \delta \int (\kappa - G) \sqrt{-g} \ dX = 0, \]  

(6.9)

under some restriction and, where \( \sqrt{-g} g_{\cdot\cdot} \) only is varied. However, we should note that this is not the sole possibility.

The first extension given by Einstein himself was by a conformal mapping of the space structure. By considering the metric space having the fundamental form

\[ ds^2 = \tilde{g}_{\cdot\cdot} \ dx^\cdot \ dx^{\cdot} \]  

where

\[ \tilde{g}_{\cdot\cdot} = e^{2\sigma} \ g_{\cdot\cdot} \]

we get

\[ K_{\cdot\cdot} - \frac{1}{2} K \ g_{\cdot\cdot} + \wedge g_{\cdot\cdot} = \tilde{g}_{\cdot\cdot} + 2 \sigma_{\cdot\cdot} \]  

(6.10)\(^1\)

\(^1\) cf. reference (26), pp. 89 - 90.
where 
\[ \bar{\bar{g}}_{\mu\nu} = \bar{\bar{K}}_{\mu\nu} - \frac{1}{2} \bar{\bar{K}} \bar{\bar{g}}_{\mu\nu} \, , \]
\( \bar{\bar{K}}_{\mu\nu} \) denoting the contracted curvature tensor derived from the new metric tensor \( \bar{\bar{g}}_{\mu\nu} \), and
\[ \Lambda = \frac{1}{2} \Delta \Sigma - 4 \Delta \Omega \]
\[ \Gamma_{\mu\nu} = 2 (\nabla_{\mu} \Omega)(\nabla_{\nu} \Omega) , \]
\[ \nabla_{1} \Sigma = g^{\kappa\lambda}(\nabla_{\mu})\nabla_{\nu} \Omega , \quad \nabla_{2} \Sigma = g^{\lambda\rho}(\nabla_{\mu})\nabla_{\nu} \Omega . \]

If the right-hand side of (6.10) is compared with the material energy tensor, the equation has the structure of Einstein's extended field equation, \( \Lambda \) playing the part of the cosmological constant.

### 6.2. A non-Einsteinian régime

The Einsteinian formulae have been obtained by lumping together the microscopic physical structure into the material energy tensor \( G_{\mu\nu} \) and/or the cosmic term. The meaningfulness of such lumping is attested to by the non-divergence of the material energy tensor. A non-Einsteinian régime may be brought about if there are physical phenomena in which the microscopic physics could be lumped into something else than \( G_{\mu\nu} \) or

\[ M_{\mu\nu} \overset{\text{def}}{=} G_{\mu\nu} + \Lambda g_{\mu\nu} = \bar{\bar{G}}_{\mu\nu} + 2 \bar{\bar{Q}}_{\mu\nu} \quad (6.11) \]

preserving the non-divergence with respect to the covariant differentiation \( \nabla_{\kappa} \), which we shall call "\( \bar{\bar{G}} \)-non-divergence".

From the point of view of physics, the Einsteinian and non-Einsteinian lumpings cannot entirely be independent of each other because there would be no physics if there were no material-energy and one of the alternative at least must carry the material-energy. The restriction so imposed lets one of the two kinds vanish whenever the other does. The latter, which is more basic, may not, even if the former does. Therefore, we shall assume

\[ N^{\rho\nu} = - x^{\rho\nu} \cdot M_{\nu\rho} \quad (6.12) \]
to be the alternative physical tensor into which the microstructural structures can be mapped. The choice between the alternative depends on the manner of observation.

We can prove that many $\hat{V}$-non-divergent tensors of rank 2 associated with $M_{\mu \nu}$ in the above manner are obtained by different lumpings. The tensor $r^{\kappa \lambda \nu}$ may or may not be a zero or unit tensor according to the manner of lumping.

Whatever it is, the condition of $\hat{V}$-non-divergence requires
\[ \hat{\nabla}_{\lambda} N^{\lambda \gamma} = 0 \tag{6.13} \]
or
\[ (\hat{\nabla}_{\lambda} M_{\mu \nu}) r^{\kappa \lambda \mu \nu} + M_{\mu \nu} \hat{\nabla}^{\kappa \lambda \mu \nu} = 0 \tag{6.13.1} \]
In order that the condition (6.13.1) is satisfied for an arbitrary $M_{\mu \nu}$ which is a symmetric tensor, the coefficient tensor $r^{\kappa \lambda \mu \nu}$ needs to be either

1) anti-symmetric with respect to $\kappa$ and $\nu$

or

2) $\hat{\nabla}_{\lambda} r^{\kappa \lambda \mu \nu} = 0 \tag{6.14}$
\[ r^{\kappa \lambda \mu \nu} (\hat{\nabla}_{\lambda} M_{\mu \nu}) = 0 \tag{6.15} \]

However, 1) is trivial since it gives $N^{\gamma \nu} = 0$. Hence the anti-symmetric part of $r^{\kappa \lambda \mu \nu}$ regarding $\kappa$ and $\nu$ is excluded so that we can assume
\[ r^{\kappa \lambda \mu \nu} = r^{(\kappa | \lambda \mu | \nu)} \tag{6.16} \]

The possible new non-divergent tensor $N^{\gamma \nu}$ is not perfectly free but is restricted by the conditions (6.14), (6.15) and (6.16) imposed upon $r^{(\kappa | \lambda \mu | \nu)}$. Equation (6.14) means that $r^{(\kappa | \lambda \mu | \nu)}$ is $\hat{V}$-non-divergent with respect to $\kappa$ and $\hat{V}$-covariant with respect to the other indices. Equation (6.15) will introduce further restrictions into the structure of $r^{(\kappa | \lambda \mu | \nu)}$, such as follows from
\[ \hat{\nabla}_{\lambda} M^{\kappa \nu} = 0, \tag{6.8.1} \]
for an arbitrary $M^\kappa\nu$. satisfying this, else $M^\kappa\nu$ itself should be covariant constant.

In order that (6.15) is reduced to (6.3.1) it is sufficient that

$$r^{\kappa\nu\lambda} \tilde{V}_\lambda$$

is proportional to

$$a \tilde{V}^\kappa \tilde{V}_\lambda + b \tilde{V}^\lambda \tilde{V}_\kappa = 2 \frac{\kappa}{\nu} + b \tilde{V}^\nu$$

where $a$ and $b$ are arbitrary coefficients. Considering the symmetry regarding $\kappa$ and $\nu$, we therefore have

$$r^{\kappa\lambda\nu\mu} = g^{(\kappa\lambda\nu\mu)} \frac{1}{2} (g^{\kappa\nu} p^{\mu\lambda} + g^{\kappa\lambda} p^{\nu\mu} ) \quad (6.17)$$

where $p^{\mu\nu}$ is an arbitrary tensor such that (6.14) is satisfied, i.e.,

$$g^{(\kappa\lambda\nu\mu)} \tilde{V}_\lambda = \tilde{V}_\lambda + \frac{\kappa}{\nu} p^{\mu\nu} = 0 \quad (6.18)$$

or

$$\tilde{V}(\kappa \tilde{V}^\nu \nu) = 0 \quad (6.18.1)$$

**Lemma 6.1.** With an arbitrary distribution of the Einsteinian material-energy tensor, there is associated, at least, a set of $\tilde{V}$-non-divergent tensors

$$N^{\kappa\nu} = g^{(\kappa\nu\mu)} p^{\nu\mu} \quad (6.19)$$

where $p^{\mu\nu}$ is an arbitrary tensor satisfying

$$\tilde{V}(\nu \nu) = 0 \quad .$$

If $N^{\kappa\nu}$ is restricted to be symmetric, we have

$$r^{\kappa\lambda\nu\mu} = g^{(\kappa\lambda\nu\mu)}$$

$$= \frac{1}{4} (g^{\kappa\nu} p^{\mu\lambda} + g^{\kappa\lambda} p^{\nu\mu} + g^{\nu\lambda} p^{\kappa\mu} + g^{\nu\kappa} p^{\lambda\mu}) \quad (6.20)$$

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where \( p^{\lambda \kappa} \) should be restricted by (6.14) which gives
\[
\frac{\mathbf{V}_\lambda}{\mathbf{V}_\lambda} \mathbf{r}^{\lambda \kappa \rho \nu} = \frac{1}{4} (g^{\nu \rho} \frac{\mathbf{V}_\lambda}{\mathbf{V}_\lambda} p^{\lambda \kappa} + g^{\nu \rho} \frac{\mathbf{V}_\lambda}{\mathbf{V}_\lambda} p^{\lambda \kappa} )
\]
\[
= \frac{1}{2} g \left( \frac{\mathbf{V}_\lambda}{\mathbf{V}_\lambda} p^{\lambda \kappa} \right) = 0.
\]
There are \( n^2 (n-1)/2 \) such equations to be satisfied by \( \mathbf{V}_\lambda p^{\lambda \kappa} \). Therefore,
\[
\frac{\mathbf{V}_\lambda}{\mathbf{V}_\lambda} p^{\lambda \kappa} = 0. \tag{6.21}
\]

On the other hand, inserting (6.20) in to (6.15), we get
\[
(\frac{\mathbf{V}_\lambda}{\mathbf{V}_\lambda} M^{\lambda \nu}) r^{\lambda \kappa \rho \nu} = \frac{1}{4} (p^{\lambda \rho} \frac{\mathbf{V}_\lambda}{\mathbf{V}_\lambda} M^{\lambda \nu} + p^{\lambda \rho} \frac{\mathbf{V}_\lambda}{\mathbf{V}_\lambda} M^{\lambda \nu} )
\]
\[
+ g^{\nu \rho} p^{\lambda \kappa} \frac{\mathbf{V}_\lambda}{\mathbf{V}_\lambda} M^{\lambda \nu} + g^{\nu \rho} p^{\lambda \kappa} \frac{\mathbf{V}_\lambda}{\mathbf{V}_\lambda} M^{\lambda \nu} )
\]
\[
= \frac{1}{4} (g^{\nu \rho} p^{\lambda \kappa} + g^{\nu \rho} p^{\lambda \kappa} ) \frac{\mathbf{V}_\lambda}{\mathbf{V}_\lambda} M^{\lambda \nu}
\]
\[
= \frac{1}{2} p^{\lambda \kappa} \frac{\mathbf{V}_\lambda}{\mathbf{V}_\lambda} M^{\lambda \nu} = 0. \tag{6.22}
\]

Hence \( p^{\lambda \kappa} \) is defined by
\[
\frac{\mathbf{V}_\lambda}{\mathbf{V}_\lambda} p^{\lambda \kappa} = 0 ; (6.21) \quad \text{and} \quad \frac{\mathbf{V}_\lambda}{\mathbf{V}_\lambda} M^{\lambda \nu} = 0 \tag{6.22}
\]

while we also have
\[
\frac{\mathbf{V}_\lambda}{\mathbf{V}_\lambda} g^{\nu \rho} = 0 , \frac{\mathbf{V}_\lambda}{\mathbf{V}_\lambda} g^{\nu \rho} M^{\lambda \nu} = 0 . \tag{6.23}
\]

A possible solution for \( p^{\lambda \kappa} \) is therefore
\[
p^{\lambda \kappa} = a g^{\lambda \kappa} \tag{6.24}
\]

where \( a \) is a constant. But this is not compulsory. (See below.) Since \( p^{\lambda \kappa} \) has not initially been restricted to be symmetric, it may apparently be generalized to
\[
p^{\lambda \kappa} = p^{(\lambda \kappa)} + p^{(\kappa \lambda)} .
\]

However, inserting this into (6.20) and the latter into (6.19), we get
\[
- N^{\lambda \nu} = g^{\lambda \kappa} p^{\nu \rho} M^{\kappa \nu}
\]
\[
= \frac{1}{4} (g^{\lambda \kappa} p^{\nu \rho} + g^{\nu \rho} p^{\lambda \kappa} + g^{\nu \rho} p^{\lambda \kappa} + g^{\nu \rho} p^{\lambda \kappa} ) M^{\kappa \nu}
\]

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\[
\phi = \frac{1}{4} (\phi^{\lambda\nu} M_{\lambda\nu} + M^{\lambda}_{\lambda} P^{\nu\lambda} + M^{\nu}_{\nu} P^{\lambda\nu} + M^{\lambda}_{\lambda} P^{\nu\lambda}) \\
= \frac{1}{2} \left\{ M^{\lambda\nu} P^{\nu\lambda} + M^{\nu}_{\nu} P^{\lambda\nu} \right\}, \\
N \overset{\text{def}}{=} N^{\lambda\nu} g_{\lambda\nu} = -\frac{1}{2} M^{\lambda\nu} P^{\nu\lambda} g_{\lambda\nu} = -\frac{1}{2} M^{\lambda\nu} P^{\nu\lambda} \\
= \frac{1}{2} M^{\lambda\nu} P^{(\nu\lambda)} \quad (6.25)
\]

Lemma 6.2. The scalar \( N = N^{\nu\rho} \delta_{\rho\lambda} \) is independent of the alternating part of \( P^{\nu\lambda} \).

Restricting the views to the symmetric part of \( P^{\nu\lambda} \) does not, however, necessarily mean (6.24). We shall show that there is at least a solution for \( P^{(\nu\lambda)} \) other than (6.24), and that it need not be \( \tilde{\nabla} \)-covariant constant. For example, let \( M_{\nu\lambda} \) be such that
\[
\tilde{\nabla}_i M_{\nu\lambda} = \tilde{\nabla}_i M_{\nu\lambda} = 0
\]
and
\[
\tilde{\nabla}_i M_{\nu\lambda} \neq 0,
\]
where
\[
P \neq 1
\]
i.e. it depends covariantly on \( x^1 \) only.

Then
\[
P^{\nu\lambda} \tilde{\nabla}_\lambda M_{\nu\lambda} = P^{\nu\lambda} \tilde{\nabla}_i M_{\rho\lambda} (P \neq 1).
\]

Therefore, if
\[
p^{\nu\lambda} \neq 0 \quad \text{and} \quad p^{\nu\lambda} = p^{\nu\lambda} = 0,
\]
then
\[
p^{\nu\lambda} \tilde{\nabla}_\lambda M_{\nu\lambda} = p^{\nu\lambda} \tilde{\nabla}_i M_{\nu\lambda} = P_{\nu\lambda} = 0.
\]

This need not contradict (6.21) which is here reduced to
\[
\tilde{\nabla}_1 P^{\nu\lambda} = 0.
\]

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Hence all conditions are satisfied. \( \hat{\varphi}_{\alpha}^\lambda \) and \( \rho_{\alpha}^\lambda \) need not be simultaneously made diagonal the latter being required only to have the form:

\[
(F_{\alpha}^\lambda) = \begin{pmatrix}
\text{d} & 0 \\
0 & \tau_{\alpha}
\end{pmatrix}; \quad (\tau, \rho \neq 1).
\]

Sometimes \( p_{\alpha}^\lambda \) can be entirely unrestricted except by (6.21). In fact, if \( M_{\lambda\nu} \) is \( V \)-covariant constant

\[
\hat{V}_\lambda M_{\nu\nu} = 0;
\]

(6.22) is automatically satisfied. However, the more primary condition (6.15) is automatically satisfied in such a case without restricting \( r_{\alpha\lambda\nu\nu} \) to the form (6.17). The meaning of the specific case seems to need further investigations.

Without such a restriction of \( M_{\nu\kappa} \), we have

**Lemma 6.3.** It is possible to find a \( \hat{V} \)-covariant non-uniform or uniform distribution of symmetric tensor of rank 2 connected with an arbitrary Einstein tensor by

\[
N^{\hat{\nu}^\kappa} = - r_{\alpha\lambda\nu\kappa} M_{\nu\kappa}
\]

where \( N^{\hat{\nu}^\kappa} \) is the required tensor, \( M_{\nu\kappa} \) the Einstein tensor and

\[
\Gamma^{\alpha\lambda\nu} = g^{\alpha\lambda\kappa}(\rho^{\nu\kappa})^{\mu^3},
\]

where \( \rho^{\nu\kappa} = \rho^{\nu}_{\alpha\lambda\nu\kappa} \), \( \hat{V}_\nu \rho^{\nu\kappa} = 0 \), and \( \rho^{\alpha\lambda} \hat{V}_\kappa M_{\nu\nu} = 0 \).

For the restricted case, we have

**Lemma 6.4.** It is possible to find a \( \hat{V} \)-covariant non-divergent distribution of symmetric tensor of rank 2 connected with an arbitrary \( \hat{V} \)-covariant constant Einstein tensor by

\[
N^{\hat{\nu}^\kappa} = - r_{\alpha\lambda\nu\kappa} M_{\nu\kappa}
\]

provided

\[
r_{\alpha\lambda\nu\kappa} = r_{(\kappa\lambda\nu\kappa)}^{(\nu\kappa)}; \quad \hat{V}_\nu r_{\alpha\lambda\nu\kappa} = 0.
\]
From Lemmas 6.3 and 6.4, we conclude:

**THEOREM 4.** There can be physically meaningful non-divergent symmetric tensors of rank 2, $N^{\mu\nu}$ having the form

$$N^{\mu\nu} = -r^{\mu\nu} \left( K_{\nu\kappa} - \frac{1}{2} \frac{\partial}{\partial x^\kappa} \delta_{\nu\mu} \right)$$

where

$$r^{\mu\nu} = r^{\mu\nu}(p, x, t)$$

and

$$\nabla_{\lambda} r^{\mu\nu} = 0.$$

$r^{\mu\nu}$ can in particular be a multiple of $g^{\mu\nu}$ where $P^{\mu\nu}$ is non-divergent and not necessarily a multiple of the metric fundamental tensor $g^{\mu\nu}$.

If the minor or microscopic structures are neglected, we shall have, from THEOREM 4, equations similar to (6.5), (6.6) etc. above, leading first to the integrated equation

$$\int \left[ k^{\mu\nu} (K_{\nu\kappa} - \frac{1}{2} \frac{\partial}{\partial x^\kappa} \delta_{\nu\mu}) + N^{\mu\nu} \right] g_{\kappa\lambda} \sqrt{-g} \ dX = 0. \quad (6.26)$$

Let the fundamental tensor $g_{\mu\nu}$ be varied. If the tensor density

$$T'_{\mu\nu} = \sqrt{-g} N^{\mu\nu},$$

which summarizes the macroscopic structures, does not change with $g_{\mu\nu}$, then

$$\delta \int g_{\mu\nu} N^{\mu\nu} \sqrt{-g} dX = \int (\delta g_{\mu\nu}) T'_{\mu\nu} dX. \quad (6.27)$$

From (6.26), therefore, follows

$$\delta \int h^{\mu\nu} (G_{\mu\nu} + \Lambda g_{\mu\nu}) \sqrt{-g} dX + \int (\delta g_{\mu\nu}) N^{\mu\nu} \sqrt{-g} dX = 0. \quad (6.28)$$

where

$$h^{\mu\nu} = r^{\mu\nu} g_{\mu\nu}. \quad (6.29)$$

But
\[ h_{\gamma}^{\kappa} G_{\gamma\kappa} = k^{\gamma\nu} K_{\nu\kappa} \]

where
\[ k^{\gamma\nu} = h^{\gamma\nu} - \frac{1}{2} h g^{\gamma\nu} \]

and
\[ h = h^{\mu\nu} g_{\mu\nu} \cdot \]

Hence
\[ \delta \left( (k^{\gamma\nu} K_{\gamma\nu} + \Lambda) \sqrt{-g} \, dx + \int (\delta g^{\mu\nu}) K_{\gamma\nu} \sqrt{-g} \, dx = 0 \right) \]
\[ (6.28.1) \]

where
\[ \Lambda = h \Lambda \cdot \]

The variational equation (6.28.1) gives a non-Einsteinian macroscopic physics, which will reduce to the Einsteinian if \( h^{\kappa\nu} \) or \( k^{\gamma\nu} \) is further restricted. The reader might wonder why a modification of the Einsteinian criterion, such as this is necessary. The physical meanings of those non-Einsteinian viewpoints which require (6.28.1) will be studied elsewhere. The Einsteinian criterion follows as a special case, as will be discussed in §6.3.

6.3. The Einsteinian régime. If the possibility is restricted to (6.24), everything will reduce to the Einsteinian régime. Such will be the case if there is no significant tensor other than the fundamental tensor so that we have
\[ r^{\kappa\lambda\nu\rho} = g^{\kappa\lambda\alpha} g^{\nu\rho\beta} \cdot \]  
\[ (6.30) \]

It is reached in a formal manner as follows.

We start with two fundamental postulates:
Postulate 1: The material-energy tensor field is not generally \( V \)-covariant uniform i.e. Lemma 6.3. is to be assumed so that we have (6.20):

\[ \delta \int h^{\alpha \nu} (K_{\nu \kappa} - \frac{1}{2} K g_{\nu \kappa} + \Lambda g_{\nu \kappa} + M_{\nu \kappa}) \sqrt{-g} \, dX = 0 \]

where (6.12), (6.26) and (6.29) are taken account of. Postulate 2 means that the variation concerns \( \rho^{\alpha \nu} \) while \( g_{\nu \kappa} \) is kept unvaried i.e. \( h^{\alpha \nu} \) is varied, \( g_{\nu \kappa} \) being kept unvaried. Hence

\[ \delta \int \delta h^{\alpha \nu} (K_{\nu \kappa} - \frac{1}{2} K g_{\nu \kappa} + \Lambda g_{\nu \kappa} + M_{\nu \kappa}) \sqrt{-g} \, dX = 0 \]

where

\[ \delta = \sqrt{-g} \, h^{\alpha \nu} \]

so that

\[ \boxed{K_{\nu \kappa} - \frac{1}{2} K g_{\nu \kappa} + \Lambda g_{\nu \kappa} + M_{\nu \kappa} = 0} \]  

(6.31)

which is the same as (6.10). Therefore, we claim the

\[ \text{Proposition. Einstein's general relativity theory states the simplest summary of macroscopic observations ignoring minor structures as perfectly as conceivable for human capability.} \]

\[ \text{It has been proven that} \]

\[ \text{Postulate 3. The world-line or trajectory of the material particle in the space-time of general relativity follows from the field equation.} \]

The geodesics condition

\[ \delta \int ds = 0 \]  

(6.32)

\[ 1) \] See, e.g., (27).
must therefore be the global approximation of the least square criterion (3.1). By (6.32) and (3.1), we see that

\[ \dot{\mathbf{X}}_a \mathcal{O} \mathbf{w}_a \mathbf{s} \]

is approximately replaced by

\[ \mathbf{w}_a \mathbf{s} \frac{1}{ds} \]

i.e., \( d\mathbf{X}_a \mathcal{O} \) by \( \frac{1}{ds} \), which is not unreasonable.

This is also obtained as a version of the wave-geometrical formula (3.5.2) which, under some macroscopic degenerations,

\[ ds \mathbf{S} ds = 0 \]

It leads to

\[ \mathbf{S} ds = 0 \quad (6.33) \]

provided

\[ ds \neq 0 \quad (6.34) \]

The exception (6.34) occurs only at

\[ \mathbf{w}^a = 0 \]

and at the velocity of light. Therefore, between the trivial configurations one the origin of the local space-time and the other the light velocity which is never attained by material bodies, we can always assume (6.33). For any aggregation of such elements we have

\[ \mathbf{S} \sum ds = 0 \]

The aggregation can in particular be a curve. We then get (6.32). Thus we obtain

**THEOREM 5.** Any observationally meaningful curve under a macroscopic degeneration is a geodesic of the observational Riemannian space-time.

It should be remarked that at light velocity and at
the origin of the local space-time the geodesical postulate loses its power. At these point we cannot eliminate $ds$ by dividing $\delta ds = 0$ by it so that the wave-geometrical criterion (3.5.2) needs to be retrained for light and for the microscopic phenomena. If the path of light is argued to be a null geodesic of the Riemannian manifold, it must follow as the limit from the path of a particle of mass. Notice also that the electromagnetic fields correspond to both extremities in that they are defined in the narrowest local space-time and that they are propagated with the velocity of light.

**THEOREM 6.** If the specific numerical value of the gravity constant and the internal structure of the material-energy tensor are not concerned about, the general relativity theory follows from statistical observation by restricting it to the most macroscopic scale.

One can, if need be, extend the viewpoint to more general formulae and derive them from observation in similar manners.

7. Comparison with Plasticity and Incompatibility Physics

It is simply by reducing the dimension number from 4 to 3 and excluding the time dimension that the entire structure of non-Riemannian, anholonomic plasticity statics is obtained. The details thereof have partly been studied in the recent Research Notes, [28], etc., and more will be reported in the near future. The correspondences are summarized in the following table:
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1) See references [29], [30] and also [31], as well as some more Research Notes which will presently be circulated.
The comparison would require some more remarks as follows:

1° Electromagnetic fields appears when the observational manifold is torn in specific manner, called "the electromagnetic tearing". The electromagnetic field summarizes all components of the torsion tensor $S^\kappa_{\mu \nu}$, regarding the upper index $\kappa$. It affords a new dimension associated with the electron charge $e$. The four components for $\kappa = 1, 2, 3, 4$ may suggest the possibility of decomposing it into 4 which may reveal certain internal structures resembling the edge and screw-dislocation distributions in imperfect crystals. Such internal structures of disturbed space-time manifold should be investigated. Also, they might have been included in some way in the present formalisms of quantum mechanics and unified field theory.

2° The existence of the Riemann-Christoffel curvature and torsion tensor of necessity leads to the Heisenberg uncertainty, as has been pointed out in D-VIII, [32]. Hence the space-time manifold may have basically discrete structure in analogy with crystal lattice. Whether there could be final discrete units which are more microscopic than the so-called elementary particles and, if there are, what they could be will be studied elsewhere. The locally Euclidean metric, such as was assumed in §1, was derived as a result from our investigation into a certain aspect of this problem, by which plasticity and space-time physics could be put in extremely direct correspondence.

3° As is the Riemann-Christoffel curvature tensor responsible for overoccupation or vacancy in crystallography, so is it responsible for material-energy in relativity theory. The Einsteinian field equation of the latter is directly carried over to the former as the equation for incompatibility. In deriving it, however, resort has been made to the particular characteristics of three-dimensional manifold where the numbers of independent components of the four-index curvature tensor and of the Ricci tensor coincide. This not being the case for the dimension 4, Einstein had to make a different approach and arrived at the
field equation without restriction of dimension number. Everything has been reduced to a variational criterion which we have further explained above as a natural consequence of macroscopic observation.

4° One particular significance of the modified variational formulation of §6.3 should be emphasized. It points to a most reasonable and convincing explanation of our theory of yielding which has been in our possession for 1½ years. Since a more detailed account of its derivation in terms of an extension of the general relativistic analogy will be given elsewhere, it suffices to point out, for the time being, that the proposed criterion is a natural condition of the modified variation for three dimensions in terms of the Euler-Schouten curvature tensor. The latter tensor, and hence the immersion in multi-dimensional space, is also a natural consequence, and not an artificial assumption, connected largely with the $R_{\nu \mu \lambda}^\kappa$ part of the $R_{\nu \mu \lambda}^\kappa$. In this problem, one need not be concerned about whether the basic space, having both $K_{\nu \mu \lambda}^\kappa$ and $L_{\nu \mu \lambda}^\kappa$ summarized into $R_{\nu \mu \lambda}^\kappa$, is Riemannian or has distant parallelism.

5° The multi-dimensional picture which we adopt in most cases, is based on the three- or four-dimensional Riemannian space enveloped by Euclidean space, as a special case of the anholonomic space $X_\nu^m$. In the present exposition, the generality is preserved by the variational treatment in regard to the parameters. Therefore, the immersion picture of 4° has been inherent from the beginning in the observational principle. It was the basis of D-V, (35), where an anholonomic plasticity theory was developed. We now have its version for unification of matter and space-time. The general anholonomic space has the advantage that it is sufficiently resourceful to carry the entire microscopic physics by virtue of the parameters $\gamma$ and the laws governing them are to be derived from the natural subsidiary condition, such as (3.5.2), which has been briefly touched upon in this Note but will emerge as the vehicle of the entire quantum-mechanical formalism, as has been suggested by Wave Geometry.
References


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