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FEATURE SPACES WHICH ADMIT AND DETECT INVARIANT SIGNAL TRANSFORMATIONS

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Summary

Information carried by a visual pattern is invariant under translations, rotations and so on, although the features of the pattern change under these transformations. A feature space admits a transformation group, when a pattern transformation induces a well-defined feature transformation. General forms of the feature spaces admitting various invariant groups are given. The normalization procedure can be carried out in such a feature space in the sense that the features of the normalized or standardized pattern are obtained from the features of the unnormalized pattern. A transformation group is said to be detectable, when any transformation can be identified by the change in the features. The detectability condition is given, and the feature spaces with which a three-dimensional motion of an object is detectable are given.

I. Introduction

It is important in designing a pattern recognition system or analyzing human perception to know the invariant transformations under which information contents of patterns are kept unchanged. Visual patterns, for example, are in most cases invariant under translations and rotations. These transformations form a Lie group. The problem was investigated by a classic paper[1)] by Pitts and McCulloch in relation to the neural mechanism of invariant perception. The features of patterns which are invariant under these transformations were also searched for by many authors.[2-6)] The Lie group structures of invariant transformations have also been investigated in relation to the human perception of visual patterns,[7-10)] and the neuronal mechanisms of invariant information processing have been studied.[11-13)]

Given a pattern, a set of features are measured at first in a pattern recognition system as well as in the human brain system. It is these feature signals that are processed in the system. It is, hence, important to study the effect, on the feature signals, of the invariant transformations of the original patterns. The present paper treats two problems: 1)To determine the invariant transformation from the changes in the features of a pattern under transformation. 2)Given features of a pattern, to obtain directly the features of the standardized or normalized version of the original pattern. We solve these problems in linear feature spaces.

A feature space is said to be detectable when the transformation of the original pattern can be detected from its features only. We give a necessary and sufficient condition for a linear feature space to be detectable. We give a general form of the feature space in which the Euclidean motion of an object, that induces a projective transformation of a pattern mapped on the retina, can be detected. The second problem is related to the standardization or normalization of patterns, which is carried into effect in the feature space. The feature space is said to admit the normalization in this case. We shall give a necessary and sufficient condition for a feature space to admit normalization. Examples of detectable feature spaces and admissible feature spaces are given.

II. Invariant Transformation Groups

2.1. Invariant transformations

We consider visual pattern signal drawn on a plane. Let (x, y) be Cartesian coordiantes of the plane. Then, a pattern is represented by a non-negative function s(x, y), which denotes the brightness at point (x, y) on the plane. The set S of these signals is called the signal space.

Every signal carries information. Generally, there are some transformations under which information of signals is kept invariant. We call them invariant transformations. We can consider, for example, the following transformations:

$$g_{00}(u) \;:\; s(x, y) \longrightarrow e^{u}s(x, y)$$

$$g_{10}(u) \;:\; s(x, y) \longrightarrow s(x - u, y)$$

$$g_{01}(u) \;:\; s(x, y) \longrightarrow s(x, y - u)$$

$$g_{11}(u) \;:\; s(x, y) \longrightarrow s(x\cos u - y\sin u,\; x\sin u + y\cos u)$$

$$g_{20}(u) \;:\; s(x, y) \longrightarrow e^{-u}s(e^{-u}z, y)$$

$$g_{02}(u) \;:\; s(x, y) \longrightarrow e^{-u}s(x, e^{-u}y)$$

Transformation $g_{00}(u)$ increases the brightness of a pattern s(x, y) by a factor e^{u}. Transformations $g_{10}(u)$ and $g_{01}(u)$ move patterns by u in the directions of the x- and y-axes, respectively. Transformation $g_{11}(u)$ rotates a pattern counterclockwise by an angle u, while $g_{20}(u)$ and $g_{02}(u)$ enlarge a pattern in the x- and y-directions, respectively, without changing the total amount of brightness. All of these are linear transformations of S to itself.

The above transformations generate transformation groups. For example, g_{00}, g_{10} and g_{01} generate a group consisting of all the parallel translations as well as changes in the brightness of patterns. We simply call it the translation group and denote it by G_1. The group generated by G_1 together with g_{11} includes all the translations and rotations. It is denoted by G_2. The group generated by G_2 together with g_{20} and g_{02} includes all the affine transformations of patterns (and hence dilatations). It is denoted by G_3. All of these are Lie groups.

2.2 Generators

For a pair (p, q), transformations $g_{pq}(u)$ forms a one-parameter subgroup of the Lie group. The finitesimal transformation $\bar{g}_{pq}$ of the subgroup is fined by

$$\bar{g}_{pq}s(x, y) = \frac{d}{du}\{g_{pq}s(x, y)\} \qquad (1)$$

t is known that the one-parameter subgroup is uniquely etermined by the generator $\bar{g}_{pq}$. Transformation $g_{pq}(u)$ s denoted symbolically by

$$g_{pq}(u) = \exp\{u\bar{g}_{pq}\}. \qquad (2)$$

The infinitesimal transformations are obtained in ur case as follows:

$$\bar{g}_{00} = I$$

$$\bar{g}_{10} = -\partial/\partial x,\ \bar{g}_{01} = -\partial/\partial y,$$

$$\bar{g}_{11} = x\partial/\partial y - y\partial/\partial x,$$

$$\bar{g}_{20} = -I - x\partial/\partial x,\ \bar{g}_{02} = -I - y\partial/\partial y,$$

here I is the identity operator.

It can be shown that any element $g \in G_3$ can niquely be decomposed into the following form

$$g = \exp\{u_{02}\bar{g}_{02}\}\cdot\exp\{u_{20}\bar{g}_{20}\}\cdot\exp\{u_{11}\bar{g}_{11}\}\cdot$$
$$\exp\{u_{01}\bar{g}_{01}\}\cdot\exp\{u_{10}\bar{g}_{10}\}\cdot\exp\{u_{00}\bar{g}_{00}\}.$$

e cannot change the order of factors because the roup is not commutative. Any $g \in G_3$, hence, can niquely be represented by a set of 6 numbers

$$\underline{u} = (u_{02}, u_{20}, u_{11}, u_{01}, u_{10}, u_{00})$$

o that it is denoted by

$$g = g(\underline{u}) \qquad (3)$$

This $\underline{u}$ is known as the canonical coordinates of G_3 of the second kind associated with the generators $\bar{g}_{pq}$.

2.3. Normalization of patterns

Two patterns $s(x, y)$ and $r(x, y)$ are called equivalent, when they are connected by an invariant transformation g,

$$r(x, y) = gs(x, y).$$

The invariant transformation group, therefore, partitions S into equivalence calsses. Normalization is a procedure to transform a pattern invariantly in its standard form. To this end, we define a standard form of signals by using the following linear mappings $a_{pq}(0 \geqq p + q \geqq 2)$ from S to the real values,

$$a_{pq}s = \int a_{pq}(x, y)s(x, y)dxdy, \qquad (4)$$

where we put

$$a_{pq}(x, y) = x^p y^q. \qquad (5)$$

We call a set of numbers $\underline{c} = (c_{pq})$ the deformation coordinates of the pattern. When $\underline{c}$ takes a specific value $\underline{c}^0$, the pattern is said to be in the standard form. In the present case, we put

$$c_{00}{}^0 = 1,\ c_{10}{}^0 = c_{01}{}^0 = c_{11}{}^0 = 0,$$

$$c_{20}{}^0 = 1,\ c_{02}{}^0 = 2.$$

This implies that a pattern is in its standard form when the total amount c_{00} of brightness is 1, the center of gravity (c_{10}, c_{01}) is at the origin, and the second-order moment matrix is a diagonal matrix with the first diagonal entry $c_{20} = 1$ and the second entry $c_{02} = 2$.

Let $g(\underline{u})$ be the transformation which changes a signal s to its standard form. Then,

$$a_{pq}\{g(\underline{u})s\} = c_{pq}{}^0 \qquad (6)$$

holds. This is the equation to obtain the $\underline{u}$ or $g(\underline{u})$. The pattern $g(\underline{u})s$ is called the normalized pattern or the standard form of s.

2.4. Transformations induced by motion

Let us consider the case where pattern $s(x, y)$ is a map of an object on the retina, whose coordinates are denoted by x and y. Let X, Y, Z be the coordinates of the object space. We consider a simple projective

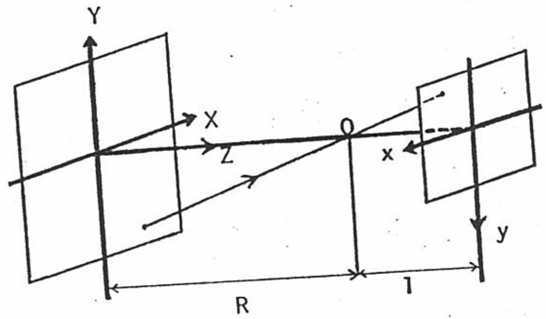

Fig.1

mapping from an object point to the retina plane through a center O (Fig. 1). Let R and l be, respectively, the distances from the center O to the origins of the object and retina spaces. Then, an object point (X, Y, Z) is mapped to a point (x, y) of the retina by

$$x = X/(R - Z),\ y = Y/(R - Z).$$

We consider a flat object put on the X-Y plane. It produces an image pattern $s(x, y)$ on the retina. The image pattern $s(x, y)$ changes as the object undergoes the Euclidean motion. The Euclidean motion includes three independent translations and three independent rotations. It is easy to see that, the translations along X-, Y- and Z- axes induce mappings $g_{10}(u)$, $g_{01}(u)$ and $g_{20}(u)\cdot g_{02}(u)$, respectively, applied to $s(x, y)$. The rotation around Z- axis induces $g_{11}(u)$. The rotations around X- and Y- axes induce, respectively

$$g_{21}(u)\ :\ s(x, y) \longrightarrow s\left(\frac{x}{1 - y\sin u}, \frac{y\cos u}{1 - y\sin u}\right),$$

$$g_{12}(u)\ :\ s(x, y) \longrightarrow s\left(\frac{x\cos u}{1 - x\sin u}, \frac{y}{1 - x\sin u}\right).$$

These are projective transformations, and their infinitesimal generators are

$$\bar{g}_{21} = -xy(\partial/\partial x) - y^2(\partial/\partial y),$$

$$\bar{g}_{12} = -x^2(\partial/\partial x) - xy(\partial/\partial y).$$

The group generated by all of these is denoted by G_m.

III. Admissibility, Detectability and Normalizability in Feature Spaces

3.1. Linear feature space

A linear feature f of a pattern s is a quantity obtained by a linear mapping m as f = ms. We treat the following type of mappings

$$f = ms = \int m(x, y)s(x, y)dxdy,$$

and call m(x, y) the measuring function of the feature f.

Let $\underline{f} = (f_1, f_2, \ldots, f_n)$ be a set of n features and let $\underline{m} = (m_1, \ldots, m_n)$ be a set of the corresponding measuring functions. An n-dimensional vector

$$\underline{f} = \underline{m}s \tag{7}$$

is called the feature signal of a pattern s, where

$$f_i = m_i s = \int m_i(x, y)s(x, y)dxdy. \tag{8}$$

Let M be the vector space spanned by n measuring functions $m_i(x, y)$. We call M the measuring space.

3.2. Admissibility

Assume that, for an invariant transformation g, there exists a linear mapping $\tilde{g}$ of the feature space such that

$$\underline{m}(gs) = \tilde{g}(\underline{m}s) \tag{9}$$

holds for any s. In this case, the feature $\underline{f}' = \underline{m}(gs)$ of the pattern gs after transformation can be obtained directly from the feature $\underline{f}$ of the original pattern by the linear transformation

$$\underline{f}' = \tilde{g}\underline{f}. \tag{10}$$

Therefore, transformation g can be carried into effect in the feature space. We say in this case that g is admissible in the feature space F or in the corresponding measuring space M. When all the elements of G are admissible, F or M is said to admit the transformation group G.

We show the condition that M admits g. Let g^* be the conjugate of g defined by

$$m(gs) = (g^*m)s. \tag{11}$$

for all s, where g^* is operated on m(x, y). The conjugates $\bar{g}^*_{pq}$ of the generators $\bar{g}_{pq}$ are as follows:

$$\bar{g}^*_{00} = I$$

$$\bar{g}^*_{10} = \partial/\partial x, \quad \bar{g}^*_{01} = \partial/\partial y$$

$$\bar{g}^*_{11} = -x(\partial/\partial y) + y(\partial/\partial x)$$

$$\bar{g}^*_{20} = x(\partial/\partial x), \quad \bar{g}^*_{02} = y(\partial/\partial y)$$

$$\bar{g}^*_{21} = y + xy(\partial/\partial x) + y(y + 2)(\partial/\partial y),$$

$$\bar{g}^*_{12} = x + xy(\partial/\partial y) + x(x + 2)(\partial/\partial x).$$

We show only one example of the derivation:

$$m\bar{g}_{10}s = \int m(-\partial/\partial x)sdx = \int \{(\partial/\partial x)m\}sdx$$
$$= (\bar{g}^*_{10}m)s,$$

which yields $\bar{g}^*_{10} = \partial/\partial x$.

Theorem 1. A necessary and sufficient condition for a feature space F or a measuring space M to admit g is that the measuring space M is closed under g^*,

$$g^*M \subset M.$$

Proof. When $g^*M \subset M$ holds, there exists $\tilde{g}_{ij}$ such that

$$g^*m_i(x, y) = \sum_{j=1}^{n} \tilde{g}_{ij}m_j(x, y). \tag{13}$$

Hence

$$f_i' = m_i gs = g^*m_i s = \Sigma\, \tilde{g}_{ij}m_j s = \Sigma\, \tilde{g}_{ij}f_j,$$

which shows that the matrix $\tilde{g} = (\tilde{g}_{ij})$ is the feature transformation corresponding to g. The converse also holds.

Let us define G^*M and $\bar{G}^*M$, respectively, by

$$G^*M = \{g^*m \mid g \in G,\ m \in M\},$$

$$\bar{G}^*M = \{\bar{g}^*m \mid \bar{g}: \text{generators of } G,\ m \in M\}.$$

Theorem 2. If an M admits all the generators of G, the M admits G itself. In other words, $\bar{G}^*M \subset M$ implies $G^*M \subset M$.

3.3. Detectability

Let $g(\underline{u})$ be the invariant transformation by which a pattern s is changed in the standard form. It is defined by

$$c_{pq}{}^0 = a_{pq}g(\underline{u})s. \tag{14}$$

When the $\underline{u}$ or the $g(\underline{u})$ can be obtained from the feature signal $\underline{f} = \underline{m}s$ without referring to the original s, the group G is said to be detectable in the feature space F or the measuring space M. Let A be the space spanned by the set of the functions $a_{pq}(x, y)$.

Theorem 3. A group G is detectable in M, when and only when

$$G^*A \subset M. \tag{15}$$

Proof. When (15) holds, we can write

$$g^*(\underline{u})a_{pq} = \sum_i h_{pqi}(\underline{u})m_i.$$

From (14), we have

$$c_{pq}{}^0 = g^*(\underline{u})a_{pq}s = \Sigma\, h_{pqi}(\underline{u})m_i s$$
$$= \Sigma\, h_{pqi}(\underline{u})f_i. \tag{16}$$

This is the equation determining $\underline{u}$ from f_i's. (We posit that the equation has a unique solution, because a_{pq}'s are independent functions responsible for invariant changes by g's: The number of the equations is the same as that of the variables $\underline{u}$.)

3.4. Normalization in the feature space

Given a pattern, we need to apply the normalization procedure to obtain the standard form of the pattern. If we can obtain the features of the standardized pattern directly from the features of the original one which is not in the standard form, the normalization can be carried into effect in the feature space. Such a feature space includes sufficient information so far as invariant transformations are concerned.

Theorem 4. The normalization can be carried into effect in the feature space, when and only when

$$G^*A \subset M,\ \bar{G}^*M \subset M. \qquad (17)$$

If $A \subset M$ holds, the condition coincides with the admissibility

$$\bar{G}^*M \subset M.$$

Proof. In order to perform normalization, we need to know $\underline{u}$ or $g(\underline{u})$ by which the given pattern is put in the standard form. This requires $G^*A \subset M$. In order to carry the normalization into effect in the feature space, the admissibility $\bar{G}^*M \subset M$ is required. Conversely, when (17) holds, we can obtain $\underline{u}$ from (16) and hence $\underline{f}' = \underline{m}g(\underline{u})s$ from (10).

IV. Feature Spaces Admitting and Detecting various G

4.1. Feature space admitting G_1

Let F_1 be a feature space admitting G_1, and M_1 be corresponding measuring space. We look for the general form of M_1.

Theorem 5. The measuring space M_1 is a space spanned by functions

$$\{x^p y^q e^{\alpha x+\beta y},\ x^p y^q e^{\bar{\alpha}x+\bar{\beta}y}\}$$

or the direct sum of such spaces, where p and q are integers satisfying

$$0 \leqq p \leqq k,\ 0 \leqq q \leqq k'$$

for some k and k', and α and β are arbitrary complex numbers where $\bar{\alpha}$ denotes the complex conjugate of α.

Proof. Let $m(x, y)$ be a function belonging to M_1. Since M_1 is closed under the generators of G_1,

$$\bar{g}^*_{10}\, m = \partial m(x, y)/\partial x$$

must be included in G_1. By the same reason, $\partial^2 m/\partial x^2$, $\partial^3 m/\partial x^3$, ..., must be included in G_1. However, M_1 should be spanned by a finite number of measuring functions, because we are considering a finite-dimensional feature space. Hence, all of the above derivatives cannot be linearly independent, so that we have for some r

$$\partial^r m/\partial x^r = \sum_{i=0}^{r-1} a_i (\partial^i/\partial x^i) m$$

where a_i are constants. The independent solutions of the above equation are given by

$$x^p e^{\alpha x},\ x^p e^{\bar{\alpha}x},\ 0 \leqq p \leqq k$$

where α is a characteristic root of the equation and k is its multiplicity. The same argument holds for $\bar{g}^*_{01}$. Combining these results, we have the theorem.

The theorem shows that a set of moment features, for example,

$$f_{pq} = \int x^p y^q s(x, y)\,dxdy, \quad 0 \leqq p \leqq k,\ 0 \leqq q \leqq k'$$

constitutes an admissible feature space ($\alpha = \beta = 0$). A set of Fourier components

$$f_{pq} = \int \exp\{ i(\alpha_p x + \beta_q y)\} s(x, y)\,dxdy$$

also constitutes an admissible feature space, where we put $k = k' = 0$ in the theorem.

4.2. Feature spaces admitting G_2 and G_3

We have the following theorem.

Theorem 6. The measuring space M_2 is spanned by functions

$$\{ x^p y^q e^{\alpha x+\beta y},\ x^p y^q e^{\bar{\alpha}x+\bar{\beta}y}\},$$

where p and q are integers satisfying $0 \leqq p + q \leqq k$ for some k. The measuring space M_3 is spanned by $\{ x^p y^q \}$, where $0 \leqq p + q \leqq k$.

Proof. Since G_1 is a subgroup of G_2, M_2 admits G_1 and hence it is a special case of M_1. Moreover, $\bar{g}^*_{11}M_2 \subset M_2$ should be satisfied. By calculating $\bar{g}^*_{11}M_1$, we obtain the former part of the theorem. Since $G_2 \subset G_3$, M_3 is a special case of M_2, satisfying $\bar{g}^*_{02}M_3 \subset M_3$, $\bar{g}^*_{20}M_3 \subset M_3$. This yields the latter half of the theorem.

We have thus proved that the moment features of orders less than a constant constitute the only type of the feature space admitting G_3, i.e., the affine transformation group. It is easy to give the procedure of obtaining the features of the normalized pattern from the features of the unnormalized pattern. The features of the normalized pattern is obviously invariant under the transformations.

4.4. Detection of motion

Let us consider the detectability $G^*A \subset M$. This implies that repeated applications of $\bar{G}^*$ on A must be included in M. Let us consider G_m induced by the Euclidean motions. It is easy to see, for example,

$$\bar{g}^*_{21}(x^p y^q) = (p + q + 1)x^p y^{q+1} + 2qx^p y^q$$

holds. This shows that $G^*_m A$ (which is obtained by repeated applications of $\bar{G}^*_m$ on A) is never included in a finite-dimensional M. Hence G_m is neither detectable nor admissible in any feature spaces.

It is, however, sometimes possible to detect the velocity of motion from the rate of change in the measured features. Let us consider a moving object, whose image $s(x, y, t)$ depends on time t. We can write

$$s(x, y, t) = g(\underline{u}(t))s, \qquad (18)$$

where $\underline{u}(t)$ represents the motion. We assume that the object is on the X-Y plane at time 0, $g(\underline{u}(0)) = I$, and consider to detect $\dot{\underline{u}}(0)$ from $\underline{f}(0)$, where $\cdot$ denotes the time derivative d/dt and

$$\underline{f}(t) = \underline{m}g(\underline{u}(t))s. \qquad (19)$$

Theorem 7. The motion velocity of G is detectable when M includes independent functions $b_r(x, y)$'s such that

$$\bar{G}^* b_r(x, y) \subset M,$$

and the number of these functions is the same as the number of the generators of the motion.

Proof. Since $b_r \in M$,

$$c_r(t) = b_r s(t) \qquad (20)$$

can be represented by a linear combination of the features f_i, where $s(t)$ denotes $s(x, y, t)$. By differentiation, we have

$$\dot{c}_r = b_r \dot{s}(t) = \sum_{p,q,i} b_r \dot{u}_{pq} \bar{g}_{pq} s(t)$$

or

$$\dot{c}_r(0) = \sum_{rpq} \tilde{g}_{pqri} f_i(0) \dot{u}_{pq}(0), \qquad (21)$$

because of

$$(d/dt)g(\underline{u}(t)) = \Sigma\, \dot{u}_{pq} \bar{g}_{pq},$$

where we put

$$\bar{g}^*_{pq} b_r = \sum_i \tilde{g}_{pqri} m_i .$$

We obtain $\dot{\underline{u}}(0)$ from $\underline{f}$ and $\dot{\underline{f}}(0)$ by solving (21).

It should be noted that, for $B = \{b_r(x, y)\}$, G^*B is in general of infinite dimensions. However, $\bar{G}^*B$ is of finite dimensions. Therefore, the detection of motion velocity may be possible even when $g(\underline{u})$ itself is not detectable. The measuring space $M = \{B, \bar{G}^*_m B\}$ generally gives an example of the motion-detectable space. We can consider, for example, the motion detectable spaces given by the following B.

case 1) $B = \{x^p y^q, \quad 0 \leqq p + q \leqq 2\}$

case 2) $B = \{e^{i(\alpha x + \beta y)}, \quad \alpha = \alpha_1, \alpha_2;\ \beta = \beta_1, \beta_2\}$

The features of the first case consist of the moments of patterns of orders less than 5, because $\bar{G}^*_m B$ consists of $x^p y^q$, $p + q \leqq 5$. The second feature space consists of Fourier components

$$S(\alpha, \beta) = \int e^{i(\alpha x + \beta y)} s(x, y)\,dxdy$$

with the derivatives up to the second order, $\partial S/\partial\alpha$, $\partial S/\partial\beta$, $\partial^2 S/\partial\alpha^2$, $\partial^2 S/\partial\beta^2$, and $\partial^2 S/\partial\alpha\partial\beta$ at points $\alpha = \alpha_1$, α_2, $\beta = \beta_1$, β_2, because $\bar{G}^*_m B$ includes these derivatives.

Conclusions

We have studied how the features of a pattern change when the pattern suffers invariant transformations. This is important, because invariant information processing should be performed based on the features only. We have proposed the concepts of the admissibility, detectability, motion-detectability and normalizability in the featrue space. All of these concerns the invariant processing of pattern signals in the feature space. The general forms of admissible, detectable and motion-detectable feature spaces are given. One of the remaining problems is to study the neural mechanisms and architectures of performing the invariant operations in the feature space.

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