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A Geometrical Theory of Moving Dislocations and Anelasticity

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INTRODUCTION

IT has recently become well recognized that one can describe the crystallographic imperfections, such as dislocations, in terms of non-Riemannian geometry. An important kind of imperfection, called the *dislocation*, has especially the character of the *torsion* in the non-Riemannian material manifold so introduced (see [1], [2], [3] etc.). Quite a number of remarkable plastic charateristics of materials have been analyzed in such an approach. It has also been clarified how, or in what relation, the conventional crystallographical dislocation theory stands to the theory of yielding constructed by Professor K. Kondo using differential geometrical terminology.[1] Here we shall attempt to elaborate a little more the geometrization of the theory of materials in order to analyze moving dislocations which cause the change of the plastic state.

It is well known that moving dislocations play an important rôle for the change of plastic state. Kondo treated the motion of a dislocation as the production of a dislocation pair and clarified its geometrical counterpart [7]. The geometrization of moving dislocation fields was first tried in a RAAG Research Note [8], on which the present paper is based.

We shall first construct a four-dimensional material manifold, which has three ordinary spatial coordinates to specify the position of the material element and a time coordinate. A metric and a Euclidean connexion will be introduced into the manifold by mapping small *material-time* pieces to small perfect *crystal-time* pieces. Generally this mapping cannot be performed as a continuous mapping but as a piecewise mapping of torn material-times. Thus, the material-time manifold can be treated as a non-Riemannian space. The torsion tensor, thus obtained, will be proved to represent the dislocation distribution as well as their motion. By classifying the components of the torsion tensor, the types of the motion of dislocations as well as the types of dislocations themselves will spontaneously be classified.

As dislocations move, the plastic state changes and so does the incompatibility distribution. The latter will be grasped as the flow of incompatibilities and we can prove the continuity of the incompatibility flow. Moreover, the anelasticity tensor, which has been introduced phenomenologically by C. Eckart [9] in order to represent the rate of the change of the natural metric, can be derived from the tensor representing the motion of dislocations.

1. Construction of Four-Dimensional Material Space

In order to obtain the geometrical terminology suitable for expressing the rôle of moving dislocations, a four-dimensional space will be constructed for a crystal including moving imperfections. It will be done in a manner quite similar to that which has been adopted in the construction of the three-dimensional non-Riemannian material space,[2] so that the space handled in the following

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1) See, e.g., [1], [4], [5], [6].
2) See, e.g., D-IX [10].

is non-Riemannian. In this paper, however, small disturbances are assumed and the distant parallelism approach is resorted to merely for simplicity's sake. A more general non-Riemannian approach is also feasible and it will afford a natural extension of the present viewpoint.

1·1. Four-dimensional material manifold. Each point of a crystal including moving imperfections can be marked by the four-dimensional Cartesian coordinates of the position-time

$$x^\kappa, \quad \kappa=0,1,2,3$$

which it occupies in real space-time, where $x^0=t$ denotes the time coordinate and $x^{\bar\kappa}$'s $(\bar\kappa=1,2,3)$ denote the space coordinates. In other words, a point x^κ $(\kappa=0,1,2,3)$ means a location $x^{\bar\kappa}$ $(\bar\kappa=1,2,3)$ in real space at time $x^0=t$. We can regard the imperfect crystal, the points of which are marked by these coordinates, as a four-dimensional manifold.

An imperfect crystal is distorted, in the sense that some strains exist in the *space* or the crystallographic axes are distorted from those of a perfect crystal, and also that each point of the crystal is moving, i. e. it changes its attribute in regard to *time*. We cannot remove all these distortions without tearing the material, for it is plastically deformed and includes dislocations.

In order to examine the distorted state of the crystal, let us tear it into an aggregate of small material-time pieces and relax each piece by removing all distortion. Then each piece becomes a perfect crystal-time. We refer to this aggregate of small relaxed material-times as being in the "natural state", and we call the tearing and relaxing process "naturalization".

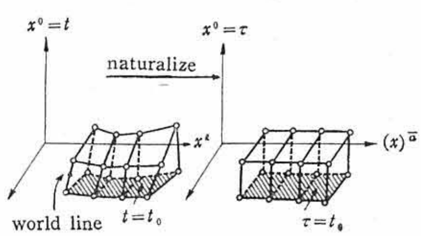

FIG. 1

With the naturalization process, an infinitesimal vector line-element dx^κ in the distorted crystal is brought to $(dx)^a$ $(a=0,1,2,3)$ in the natural state, where (a) denotes a Cartesian coordinate system (see Fig. 1). We can regard $(dx)^a$ the components of the vector dx^κ with reference to a new reference frame, which we call the "natural frame", and it is generally non-holonomic in correspondence with the tearing process.

It can be assumed that dx^κ and $(dx)^a$ are linearly related

$$\begin{aligned} dx^\kappa &= B^\kappa_a (dx)^a, \\ (dx)^a &= B^a_\kappa dx^\kappa, \end{aligned} \tag{1·1}$$

where $B^\kappa_a B^a_\lambda=\delta^\kappa_\lambda$, $B^\kappa_a B^b_\kappa=\delta^b_a$ hold. B^κ_a's are functions of x^λ. If the torn pieces in the natural state are so rearranged, then all the corresponding crystallographic axes will become parallel. This is a permissible process.[1] By this rearrangement, we can determine B^κ_a uniquely, and this means to introduce the assumption of distant parallelism.[2] A set of vectors $\mathbf{e}_a$ $(a=0,1,2,3)$ can be associated with each point of the distorted material-time as a basis of the natural frame (a). Obviously the κ-th component of $\mathbf{e}_a$ is B^κ_a. The spatial set $\mathbf{e}_{\bar a}$ $(\bar a=1,2,3)$ of $\mathbf{e}_a$'s can be determined by referring to the crystallographic axes, whereas $\mathbf{e}_0$ can be detemined by referring to the velocity $\mathbf{v}$ or the trace of the motion of the material point. The trace may be called a "world-line" in a sense slightly different from that in the theory of relativity.

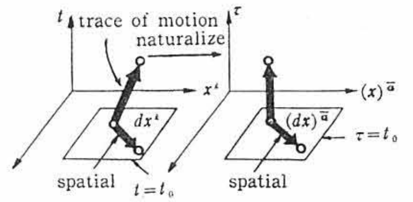

FIG. 2

Here let us determine $\mathbf{e}_a$ or the transformation tensor B^κ_a, by which the distorted state of material is represented. Splitting (1·1) into the spatial and time terms, we have

$$\left.\begin{aligned} dx^{\bar\kappa} &= B^{\bar\kappa}_{\bar a}(dx)^{\bar a}+B^{\bar\kappa}_0 d\tau, \\ dt &= B^0_{\bar a}(dx)^{\bar a}+B^0_0 d\tau, \end{aligned}\right\} \tag{1·2}$$

where $\bar\kappa$ and $\bar a$ run over the spatial coordinates alone, i.e. $\bar\kappa$, $\bar a=1,2,3$, and we put $dx^0=dt$ for the κ-frame, $dx^0=d\tau$ for the (a)-frame. We can see that a spatial vector $(dx)^a=((dx)^{\bar a}, 0)$ in the

1) This process is clarified as the perfect tearing and refurbishment [11].

2) See [10].

natural state is the image of a spatial vector $dx^{\kappa} = (dx^{\bar\kappa}, 0)$ of the deformed state (see Fig. 2). Hence putting $dt = d\tau = 0$ in (1·2), we have

$$B^0_{\bar a} = 0 \qquad (\bar a = 1, 2, 3) \qquad (1\cdot 3)$$

and we see that $B^{\bar\kappa}_{\bar a}$ represent the spatially deformed state at an instant t. Notice the assumption (1·3) which indicates that the time is not essentially affected by the naturalization or tearing, so that the special relativistic formalism is not assumed. Next, we see that a time-vector $(dx)^a = (0, 0, 0, d\tau)$ in the natural state is the image of the trace of a crystal point, i.e. the world-line of the point in the deformed state. Hence $dx^{\kappa} = (v^1 dt, v^2 dt, v^3 dt, dt)$, is mapped to the $(dx)^a$, where $v^{\bar\kappa}$ is the (spatial) velocity of the material point. Putting $(dx)^{\bar a} = 0$ in (1·2), we then have

$$dx^{\bar\kappa} = B^{\bar\kappa}_0 d\tau,$$
$$dt = B^0_0 d\tau,$$

so that

$$B^{\bar\kappa}_0 = v^{\bar\kappa}, \ (\bar\kappa = 1, 2, 3) \text{ and } B^0_0 = 1,$$

hold. Putting $B^{\kappa}_a = \delta^{\kappa}_a + \beta^{\kappa}_a$, we have

$$B^{\kappa}_a = \begin{array}{c|c|c} & \bar a & 0 \\ \hline \bar\kappa & \delta^{\bar\kappa}_{\bar a} + \beta^{\bar\kappa}_{\bar a} & v^{\bar\kappa} \\ \hline 0 & 0 & 1 \end{array}, \qquad (1\cdot 4)$$

where $\beta^{\bar\kappa}_0 = v^{\bar\kappa}$, $\beta^0_{\bar a} = 0$ and $\delta^{\bar\kappa}_{\bar a}$ is Kronecker's delta.

Since the distortion is assumed to be small, we can neglect the higher order terms of $\beta^{\bar\kappa}_{\bar a}$ and $v^{\bar\kappa}$.[1] Then the inverse transformation tensor B^b_λ can be calculated as

$$B^b_\lambda = \begin{array}{c|c|c} & \bar\lambda & 0 \\ \hline \bar b & \delta^{\bar b}_{\bar\lambda} - \beta^{\bar b}_{\bar\lambda} & -v^{\bar b} \\ \hline 0 & 0 & 1 \end{array}, \qquad (1\cdot 4\cdot 1)$$

1) The velocity $v^{\bar\kappa}$ of a lattice point is also considered to be small.

where

$$\beta^{\bar b}_{\bar\lambda} = \delta^{\bar b}_{\bar\kappa} \delta^{\bar a}_{\bar\lambda} \beta^{\bar\kappa}_{\bar a},$$
$$v^{\bar a} = \delta^{\bar a}_{\bar\kappa} v^{\bar\kappa}.$$

1·2. Metric tensor and strain tensor. Let us introduce a metric into the material manifold. We first consider the case of a perfect crystal, i.e. a crystal in the natural state. Since the frame (a) has been fixed to be orthogonal in *space*, the square of the real length ds^2 of a spatial vector $(dx)^a = ((dx)^{\bar a}, 0)$ is written as

$$ds^2 = \delta_{\bar b \bar a} (dx)^{\bar a} (dx)^{\bar b}$$
$$= (dx)^{1^2} + (dx)^{2^2} + (dx)^{3^2}$$

which may be regarded as the spatial part of a metric. In other words, we can introduce the metric tensor g_{ba} having the form

$$g_{ba} = \begin{array}{c|ccc|c} & & \bar a & & 0 \\ \hline & 1 & 0 & 0 & \\ \bar b & 0 & 1 & 0 & \\ & 0 & 0 & 1 & \\ \hline 0 & & & & \end{array}, \qquad (1\cdot 5)$$

where the components for $a, b \neq \bar a, \bar b$ are to be studied further. Owing to the arbitrary character of the extension of the metric, some assumptions may be introduced in regard to the cross components of g_{ba} relating to space and time such as $g_{0\bar b}$.

Assumption. So long as the metric characteristics alone are concerned, it is impossible to distinguish either between the positive and the negative direction of time by keeping the space-part fixed, or between the positive and negative direction for a space vector by keeping the time part fixed.

From this assumption, the vectors $((dx)^{\bar a}, d\tau)$ and $((dx)^{\bar a}, -d\tau)$ are not mutually distinguished in regard to their absolute amount. Hence we cannot but assume $g_{0\bar a} = g_{\bar b 0} = 0$. The time component g_{00} is still indefinite. Putting $g_{00} = c$, the metric tensor of the four-dimensional material space is given the structure

$$g_{ba}=\bar{b}\left\{\begin{array}{c|ccc|c} & a & \bar{a} & & 0 \\ \hline b & 1 & 0 & 0 & 0 \\ & 0 & 1 & 0 & 0 \\ & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & c \end{array}\right. \qquad (1\cdot 6)$$

for a perfect crystal. If the indefinite constant c is positive, the space is Euclidean, and if c is negative, the space is Minkowskian. In this paper, we need not determine c.

The length ds of line element dx^κ in the deformed crystal is defined by the length of the corresponding element of a perfect crystal after naturalization. Since dx^κ corresponds to $(dx)^a = B^a_\kappa dx^\kappa$, we have

$$ds^2 = g_{ba}(dx)^a(dx)^b = g_{ba}B^a_\kappa B^b_\lambda dx^\lambda dx^\kappa,$$

and the natural metric tensor can be written as

$$g_{\lambda\kappa} = g_{ba}B^a_\kappa B^b_\lambda \qquad (1\cdot 7)$$

in the κ-frame.

Substituting $(1\cdot 4\cdot 1)$ in $(1\cdot 7)$, the metric tensor is rewritten as

$$g_{\lambda\kappa}=\bar{\lambda}\left\{\begin{array}{c|c|c} & \kappa \quad \bar{\kappa} & 0 \\ \hline \lambda & \delta_{\lambda\kappa}-2\beta_{(\lambda\kappa)} & -v^\lambda \\ \hline 0 & -v^\kappa & c \end{array}\right. \qquad (1\cdot 8)$$

where $\beta_{\lambda\kappa}=\beta_\lambda{}^a\delta_{a\kappa}$ and $\beta_{(\lambda\kappa)}$ is the symmetric part of $\beta_{\lambda\kappa}$, i. e. $\beta_{(\lambda\kappa)}=\frac{1}{2}(\beta_{\lambda\kappa}+\beta_{\kappa\lambda})$.

The local characteristics of the distorted state, i. e. the distorted state of the neighbourhood of a point of the crystal, is represented by the difference between the squared natural length $ds^2=g_{\lambda\kappa}dx^\kappa dx^\lambda$ and the squared distorted length $\tilde{ds}^2=\tilde{\delta}_{\lambda\kappa}dx^\lambda dx^\kappa$, where

$$\tilde{\delta}_{\lambda\kappa}=\begin{array}{c|ccc|c} & \lambda & \bar{\lambda} & & 0 \\ \hline \kappa & 1 & 0 & 0 & 0 \\ \bar{\kappa} & 0 & 1 & 0 & 0 \\ & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & c \end{array}\,.$$

Therefore, we define the strain tensor $\varepsilon_{\lambda\kappa}$ by

$$\varepsilon_{\lambda\kappa} \overset{\text{def}}{=} \frac{1}{2}(\tilde{\delta}_{\lambda\kappa}-g_{\lambda\kappa}), \qquad (1\cdot 9)$$

which shows the local distortion of the material. Using $(1\cdot 8)$ we have

$$\varepsilon_{\lambda\kappa}=\bar{\lambda}\left\{\begin{array}{c|c|c} & \kappa \quad \bar{\kappa} & 0 \\ \hline \lambda & \beta_{(\lambda\kappa)} & \frac{1}{2}v^\lambda \\ \hline 0 & \frac{1}{2}v^\kappa & 0 \end{array}\right. , \qquad (1\cdot 10)$$

of which the spatial components $\varepsilon_{\lambda\kappa}=\beta_{(\lambda\kappa)}$ exactly coincide with those of the ordinary strain tensor defined in the three-dimensional material manifold.

1·3. Metric affine connexion with distant parallelism. In order to investigate more macroscopic characteristics of the distortion than is shown by the metric or strain tensor, we introduce an affine connexion by which the relation between the distorted states around two neighbouring points will be clarified. This is closely connected with the non-holonomic character of the natural frame (a).

An affine connexion can be introduced by defining parallelism between two vectors p^κ and $p^\kappa+dp^\kappa$ located respectively at a point $A(x^\lambda)$ and at a neighbouring point $A'(x^\lambda+dx^\lambda)$. We define the two vectors as parallel, when they are parallel in the ordinary sense after being mapped on the perfect crystal in the natural state. In this sense, all the corresponding crystallographic axes are parallel and all the lines representing the traces of the motions of crystallographic points are parallel, i. e. all the world-lines are parallel. If we construct a four-dimensional lattice from the world-picture of a three-dimensional lattice by quantizing the time, we may say that all the corresponding lattice axes are parallel.

Since the ordinary parallelism holds in the map corresponding to the perfect tearing, we put

$$\Gamma^a_{cb}=0 \qquad (1\cdot 11)$$

in the natural frame (a). Since the relation

$$\Gamma_{[cb]}^{\ a} = S_{cb}^{\cdot\cdot a} - \Omega_{cb}^{\ a}$$ [1]

holds in regard to any non-holonomic frame, where $\Omega_{cb}^{\ a}$ is the non-holonomic object defined by

$$\Omega_{cb}^{\ a} \overset{\text{def}}{=} B_{[c}^{\mu} B_{b]}^{\kappa} \partial_\mu B_\kappa^a,$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu},$$

the torsion tensor $S_{cb}^{\cdot\cdot a}$ is defined by (1·11) to be equal to the non-holonomic object of the frame (a):

$$S_{cb}^{\cdot\cdot a} \overset{\text{def}}{=} \Omega_{cb}^{\ a}. \tag{1·12}$$

From (1·11), we have

$$\Gamma_{\mu\lambda}^{\kappa} = B_b^\kappa \partial_\mu B_\lambda^b \tag{1·13}$$

and

$$S_{\mu\lambda}^{\cdot\cdot\kappa} = B_b^\kappa \partial_{[\mu} B_{\lambda]}^b, \tag{1·14}$$

which is rewritten as

$$S_{\mu\lambda}^{\cdot\cdot\kappa} = -\partial_{[\mu} \beta_{\lambda]}^{\kappa}. \tag{1·15}$$

Obviously this space has distant parallelism, i. e. the Riemann-Christoffel curvature tensor $R_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa}$ defined by

$$R_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa} = 2\partial_{[\nu}\Gamma_{\mu]\lambda}^{\kappa} + 2\Gamma_{[\nu|\rho|}^{\kappa}\Gamma_{\mu]\lambda}^{\rho}$$

vanishes identically

$$R_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa} = 0.$$

Hence all the plastic imperfections are summarized in the torsion tensor in this case of the perfect tearing.

2. Representation of Moving Dislocations by Torsion Field

It will be shown that continuously distributed *moving* dislocations are represented by the torsion tensor field. All the components of the torsion tensor have their corresponding physical counterparts, one showing edge and screw dislocations, another showing their motions, etc. We shall classify the types of dislocations and their motions by the components of the torsion tensor. Identities satisfied by these components will be derived, which can be compared with the equations of electromagnetism.

2·1. Classification of moving dislocations by the components of the torsion tensor. Since all imperfections are represented by the torsion tensor in the theory of distant parallelism, the physical meaning of the torsion must be clarified first. When the torsion tensor $S_{\mu\lambda}^{\cdot\cdot\kappa}$ exists, one fails to construct a parallelogram in space, as follows. Let us consider two vectors at A, $\overrightarrow{AB}(\underset{1}{dx^\nu})$ and $\overrightarrow{AC}(\underset{2}{dx^\nu})$, and let $\overrightarrow{CD} \mathrel{\underline{/\!/}} \overrightarrow{AB}$, $\overrightarrow{BE} \mathrel{\underline{/\!/}} \overrightarrow{AC}$ (see Fig. 3, where $\overrightarrow{AB} \mathrel{\underline{/\!/}} \overrightarrow{CD}$ means that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel and the lengths are equal). Generally, D and E do not coincide, and the discrepancy $\overrightarrow{DE}(\Delta x^\lambda)$ is expressed by

$$\Delta x^\lambda = -S_{\nu\mu}^{\cdot\cdot\lambda} \underset{1}{dx^{[\nu}} \underset{2}{dx^{\mu]}}. \tag{2·1}$$ [2]

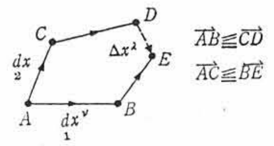

FIG. 3

The physical rôles of the torsion is embodied by dislocations such as shown in Fig. 4 and the discrepancy Δx^λ is called the Burgers vector. Hence one may call the torsion tensor the dislocation (density) tensor.

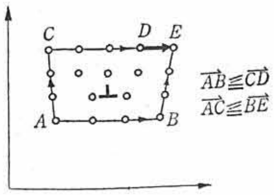

FIG. 4

The types of dislocations and their motions can be classified into four groups by means of the types of the components of the torsion tensor. From (1·15), we have

1) [ ] means "alternation", for instance,

$$\Gamma_{[cb]}^{\ a} = \frac{1}{2}(\Gamma_{cb}^{\ a} - \Gamma_{bc}^{\ a}).$$

2) See, e.g., [12].

$$S_{\mu\lambda}^{\cdot\cdot\,0}=0$$

and

$$S_{\mu\lambda}^{\cdot\cdot\,\kappa}=-S_{\lambda\mu}^{\cdot\cdot\,\kappa}. \qquad (2\cdot2)$$

Therefore, the different types are represented by the following four components,

$$S_{12}^{\cdot\cdot\,1},\ S_{12}^{\cdot\cdot\,3},\ S_{01}^{\cdot\cdot\,2},\ S_{01}^{\cdot\cdot\,1}.$$

The first two are the purely spatial components and the latter two are the mixed components. It is easy to see that the spatial components represent edge and screw dislocations.[1]

i) $S_{12}^{\cdot\cdot\,1}$: *edge dislocation; the Burgers vector lying in the x^1-direction, the dislocation line running along the x^3-axis (which is perpendicular to the x^1-x^2 plane, if the axes are orthogonal* —Fig. 5).

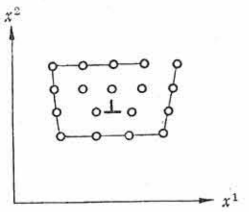

FIG. 5

ii) $S_{12}^{\cdot\cdot\,3}$: *screw dislocation; the Burgers vector lying in the x^3-direction, the disocation line running along the x^3-axis* (Fig. 6).

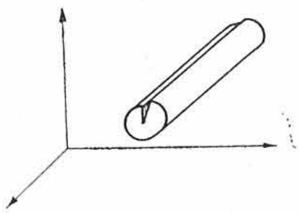

FIG. 6

Let us here consider the physical meaning of the mixed components. We can examine it from the projective viewpoint [13],[2] in which one of the fundamental points moves along the parameter line of x^0. It has been shown in D-XI [7] that the motion of an edge dislocation, say $S_{21}^{\cdot\cdot\,2}$, along the direction of the Burgers vector is a slip comparable with the appearance of Orowan's pair. Now that we replace the lower index 2 by 0, the time in which the glide takes place is measured by the length of glide along the direction of x^2,

$$U dt = dx^2$$

where U is the velocity of the dislocation. This is the origin of $S_{01}^{\cdot\cdot\,2}$ from the glide motion of $S_{21}^{\cdot\cdot\,2}$. On the other hand, the climb of $S_{12}^{\cdot\cdot\,2}$ is expressed by $S_{02}^{\cdot\cdot\,2}$ in which the time of climb is measured by the length of climb along the direction of x^1. In a manner simlar to this, we see that the slip of $S_{31}^{\cdot\cdot\,2}$ in the x^3-direction is also represented by $S_{01}^{\cdot\cdot\,2}$.

From the above considerations, we can expect that the motion of a dislocation $S_{\mu\lambda}^{\cdot\cdot\,\kappa}$ with velocity U^μ is represented by the mixed type component $S_{0\lambda}^{\cdot\cdot\,\kappa}$ of the torsion tensor, satisfying

$$S_{0\lambda}^{\cdot\cdot\,\kappa}=-S_{\mu\lambda}^{\cdot\cdot\,\kappa}U^\mu. \qquad (2\cdot3)^{3)}$$

We shall prove this.

Theorem 1. *The motion of dislocation $S_{\mu\lambda}^{\cdot\cdot\,\kappa}$ moving with velocity U^μ gives rise to*

$$S_{0\lambda}^{\cdot\cdot\,\kappa}=-U^\mu S_{\mu\lambda}^{\cdot\cdot\,\kappa}. \qquad (2\cdot4)$$

Proof: Let us consider a dislocation field $S_{\mu\lambda}^{\cdot\cdot\,\kappa}(x^\rho)$ at $t=0$. When it is moving with the velocity U^ρ, the moving dislocation field can be expressed by the tensor function of $x^\rho-U^\rho t$,

$$S_{\mu\lambda}^{\cdot\cdot\,\kappa}(x^\rho,t)=S_{\mu\lambda}^{\cdot\cdot\,\kappa}(x^\rho-U^\rho t). \qquad (2\cdot5)$$

From (1·15), we see that

$$\partial_{[\nu}S_{\mu\lambda]}^{\cdot\cdot\,\kappa}=0$$

identically holds. Taking account of (2·2) it is transformed to

1) See [1], [2], [3], [10], etc.
2) This point of view was suggested to the author by Prof. Kondo. See also D-VII [14].

3) It should be noted that tensor components of mixed type such as $S_{0\mu\lambda}$, $K_{0\nu\mu\lambda}$ are not invariant under Galilei transformations. The Galilei invariant components are those whose contravariant indices are fixed to 0 and whose covariant indices are fixed to the space part. Hence, $S^{0}_{\cdot\mu\lambda}$ and $K^{0}_{\cdot\nu\mu\lambda}$ are Galilei invariant quantities, which represent the plastic defects moving in the material. They are not generally the same as $S_{0\mu\lambda}$ and $K_{0\nu\mu\lambda}$ which include the ones moving with the material. However, in our linear theory for small disturbances, their distinction is disregarded.

$$\partial_\nu S_{\mu\lambda}^{\cdot\cdot\kappa} = 2\partial_{[\mu} S_{|\nu|\lambda]}^{\cdot\cdot\kappa}. \quad (2\cdot6)$$

Putting $\nu=0$, $\mu=\bar{\mu}$, $\lambda=\bar{\lambda}$, $\kappa=\bar{\kappa}$, we obtain

$$\partial_t S_{\bar{\mu}\bar{\lambda}}^{\cdot\cdot\bar{\kappa}} = 2\partial_{[\bar{\mu}} S_{|0|\bar{\lambda}]}^{\cdot\cdot\bar{\kappa}},$$

where

$$\partial_t = \partial_0.$$

By differentiating (2·5), it follows that

$$\partial_t S_{\bar{\mu}\bar{\lambda}}^{\cdot\cdot\bar{\kappa}} = -U^{\bar{\rho}}\partial_{\bar{\rho}} S_{\bar{\mu}\bar{\lambda}}^{\cdot\cdot\bar{\kappa}}.$$

Using (2·6), it is further transformed to

$$\partial_t S_{\bar{\mu}\bar{\lambda}}^{\cdot\cdot\bar{\kappa}} = -2\partial_{[\bar{\mu}} S_{|\bar{\rho}|\bar{\lambda}]}^{\cdot\cdot\bar{\kappa}} U^{\bar{\rho}},$$

which verifies (2·3).

From this theorem, we can clarify the following meaning of the mixed components:

iii) $S_{0\bar{1}}^{\cdot\cdot\bar{2}}$: *glide of dislocation; two kinds of motions are represented by* $S_{0\bar{1}}^{\cdot\cdot\bar{2}}$, *one corresponding to the motion of* $S_{\bar{3}\bar{1}}^{\cdot\cdot\bar{2}}$ *in the* x^3*-direction, and the other to the motion of* $S_{\bar{2}\bar{1}}^{\cdot\cdot\bar{2}}$ *in the* x^2*-direction.*

iv) $S_{0\bar{1}}^{\cdot\cdot\bar{1}}$: *climb of dislocation; the motion of* $S_{\bar{2}\bar{1}}^{\cdot\cdot\bar{1}}$ *in the* x^2*-direction or the motion of* $S_{\bar{3}\bar{1}}^{\cdot\cdot\bar{1}}$ *in the* x^3*-direction.*

The above meaning can also be ascertained by the following physical considerations. Let us consider the material line-element $\overrightarrow{AB}$ represented by the vector $(dx^{\bar{\mu}})$. After a small time dt, A and B change to A' and B', respectively (see Fig. 7). Since $\overrightarrow{AA'}$ and $\overrightarrow{BB'}$ represent the world-lines of the material points A and B, respectively, they are parallel, i.e., $\overrightarrow{AA'} \;/\!\!/\; \overrightarrow{BB'}$. On the other hand, $\overrightarrow{AB}$ and $\overrightarrow{A'B'}$ are not necessarily parallel by virtue of the torsion of the space. Let $\overrightarrow{A'C} \;/\!\!/\; \overrightarrow{AB}$, then the discrepancy $\overrightarrow{CB'}$ is represented by

$$\Delta x^{\bar{\lambda}} = -\frac{1}{2} S_{0\bar{\mu}}^{\cdot\cdot\bar{\lambda}} dx^{\bar{\mu}} dt.$$

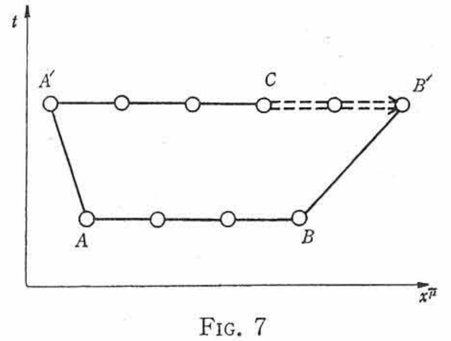

FIG. 7

Since $\overrightarrow{AB}$ and $\overrightarrow{A'C}$ are parallel and have equal length in the natural state, we can say $\overrightarrow{AB}$ is plastically changed to $\overrightarrow{A'B'}$ by $\Delta x^{\bar{\lambda}}$.

Here we take $\overrightarrow{AB}$ parallel to the x^1-axis. Then we see that the component $S_{0\bar{1}}^{\cdot\cdot\bar{1}}$ represents the increment or decrement of the number of the atoms contained in the line-element $(dx^1, 0, 0, 0)$ after a small time dt. This means that vacancies or interstitial masses appear in the line-element. This is a material non-conservative motion of dislocation and it has been pointed out that a climb of dislocations can be responsible for such a case.

On the other hand, $S_{0\bar{1}}^{\cdot\cdot\bar{2}}$ shows that one end point of the line element constituting a vector $(dx^1, 0, 0, 0)$ (the atom B in Fig. 7) is plastically displaced in the x^2-direction after a small time dt, while the other end of the element (the atom A) is not. The slip of dislocations causes the plastic motion of this type. After this slip, atoms above the slip surface are displaced in the direction of the Burgers vector, while the atoms under it are not (see Fig. 8).

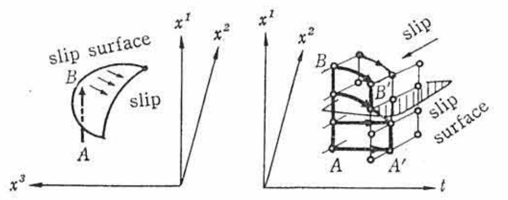

FIG. 8

2·2. Identities satisfied by the dislocation tensor. From the definition of the torsion or dislocation tensor, we have

$$S_{\mu\lambda}^{\cdot\cdot\kappa} = -\partial_{[\mu}\beta_{\lambda]}^{\kappa}, \quad (2\cdot7)$$

where $\beta_0^{\bar{\kappa}} = v^{\bar{\kappa}}$, $\beta_0^{\,0} = \beta_{\bar{\lambda}}^{\,0} = 0$ and $\beta_{\bar{\lambda}}^{\bar{\kappa}}$ is the so-called asymmetric strain tensor. Splitting this equation into the spatial and the mixed components, we have the relations

$$\left.\begin{aligned} S_{\bar{\mu}\bar{\lambda}}^{\cdot\cdot\bar{\kappa}} &= -\partial_{[\bar{\mu}}\beta_{\bar{\lambda}]}^{\bar{\kappa}}, \\ S_{0\bar{\lambda}}^{\cdot\cdot\bar{\kappa}} &= \frac{1}{2}(\partial_{\bar{\lambda}} v^{\bar{\kappa}} - \partial_t \beta_{\bar{\lambda}}^{\bar{\kappa}}), \end{aligned}\right\} \quad (2\cdot8)$$

among the torsion (dislocation and its motion), the velocities of points and asymmetric strain tensor.

From (2·7), we have the identity

$$\partial_{[\nu} S_{\mu\lambda]}^{\cdot\cdot\kappa} = 0$$

or

$$\partial_\tau(\epsilon^{\tau\nu\mu\lambda}S_{\mu\lambda}^{\cdot\cdot\kappa})=0,$$ [1]

which represents the non-divergent character of the dislocations in the four-dimensional space. This relation splits into

$$\left.\begin{aligned}&\partial_{[\nu}S_{\mu\lambda]}^{\cdot\cdot\kappa}=0\quad(\partial_\nu(\epsilon^{\nu\mu\lambda}S_{\mu\lambda}^{\cdot\cdot\kappa})=0),\\&\partial_t S_{\mu\lambda}^{\cdot\cdot\kappa}=2\partial_{[\mu}S_{|0|\lambda]}^{\cdot\cdot\kappa}.\end{aligned}\right\}\quad(2\cdot9)$$

The first equation of (2·9) means the non-divergent character of the dislocation field in the three-dimensional space. The latter equation relates the changes of dislocation distribution $\partial_t S_{\mu\lambda}^{\cdot\cdot\kappa}$ to the motion of dislocations represented by $S_{0\lambda}^{\cdot\cdot\kappa}$. Such has originated from the assumption of the four-dimensional distant parallelism criterion

$$R_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa}=0.$$

It has been discussed in Research Note, No. 42 [15], that the electromagnetism is derived from the torsion tensor in a distant parallelism space of perfect tearing. Hence it is expected that $S_{\mu\lambda}^{\cdot\cdot\kappa}$ and $S_{0\lambda}^{\cdot\cdot\kappa}$ can be compared in regard to their structure with the electromagnetic fields. In fact, the first equation of (2·9) is compared with

$$\mathrm{div}\mathbf{H}=0,$$

and the second one with

$$\mathrm{rot}\mathbf{E}=\frac{\partial}{\partial t}\mathbf{H},$$

where $\epsilon^{\nu\mu\lambda}S_{\mu\lambda}^{\cdot\cdot(\kappa)}$ and $S_{0\lambda}^{\cdot\cdot(\kappa)}$ are compared with the magnetic and electric fields, H^ν and E_λ, respectively.[2]

3. Change of Plastic State and Incompatibility

Moving dislocations give rise to a change of the plastic state, and hence the incompatibility changes also with them. We shall clarify the flow of incompatibilities from the point of view of moving dislocations. The anelasticity tensor, which has been introduced by C. Eckart [9] in order to represent the rate of the change of plastic state, can be derived from the moving dislocation tensor. The meaning of the four-dimensional incompatibility tensor, as well as its relation to the anelasticity tensor, will be clarified.

3·1. Incompatibility. When the deformation does not preserve the topology, i.e. some pairs of neighbouring atoms do not remain so after deformation, it is said to be plastic. In this case we cannot define a unique displacement u_κ for each point relative to a perfect lattice. When the deformation is compatible, there exists u_κ which is related to the tensor $\varepsilon_{\lambda\kappa}$ by

$$\varepsilon_{\lambda\kappa}=\partial_{(\lambda}u_{\kappa)}.\quad(3\cdot1)$$

The integrability condition for (3·1) is, as is well known, that the curvature tensor $K_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa}$ by the Levi-Civita parallelism associated with the metric tensor $g_{\lambda\kappa}$ vanishes, i.e.

$$\begin{aligned}K_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa}&\overset{\mathrm{def}}{=}2\partial_{[\nu}\left\{{\kappa \atop \mu]\lambda}\right\}+2\left\{{\kappa \atop [\nu|\rho|}\right\}\left\{{\rho \atop \mu]\lambda}\right\}\\&\doteqdot-4\partial_{[\nu}\partial_{|[\lambda}\varepsilon_{\kappa]|\mu]}=0.\end{aligned}\quad(3\cdot2)$$

In the case of plastic deformation, (3·2) does not generally hold, and the non-zero tensor $K_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa}$ can be called the incompatibility tensor.[3]

The spatial components $K_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa}$ of the curvature or incompatibility tensor is the ordinary incompatibility which means the existence of residual strains (stresses). On the other hand, the mixed components such as $K_{0\mu\lambda}^{\cdot\cdot\cdot\kappa}$ represent that the plastic deformation is now going on. Let us investigate what character the four-dimensional incompatibility tensor has.

Bianchi's identity for small disturbances can be written as

$$\left.\begin{aligned}&\partial_{[\rho}K_{\nu\mu]\lambda}^{\cdot\cdot\cdot\kappa}=0,\\&\partial_\rho(\epsilon^{\rho\sigma\nu\mu}K_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa})=0,\end{aligned}\right\}\quad(3\cdot3)$$

which shows the non-divergent character of the incompatibility. The spatial part of (3·3) is rewritten as

$$\begin{aligned}&\partial_{[\rho}K_{\nu\mu]\lambda}^{\cdot\cdot\cdot\kappa}=0,\\&\partial_\rho(\epsilon^{\rho\nu\mu}K_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa})=0.\end{aligned}\quad(3\cdot4)$$

1) $\epsilon^{\tau\nu\mu\lambda}$ is Eddington's quantity, i.e., $\epsilon^{\tau\nu\mu\lambda}=1(-1)$, when $(\tau\nu\mu\lambda)$ is an even (odd) permutation of (0, 1, 2, 3), and is equal to 0, otherwise.

2) The equations corresponding to the other two Maxwell equations are given in D-XVII [16]. See also E-XI [17].

3) Usually,

$$I_{\mu\lambda}=K_{\mu\lambda}-\frac{1}{2}Kg_{\mu\lambda}$$

is called the incompatibility tensor, where $K_{\mu\lambda}=K_{\nu\mu\lambda}^{\cdot\cdot\cdot\nu}$, $K=K_{\mu}^{\cdot\mu}$.

The mixed components are rewritten as

$$\partial_t K_{\bar\nu\bar\mu\bar\lambda}{}^{\bar\kappa} = 2\partial_{[\bar\nu} K_{|0|\bar\mu]\bar\lambda}{}^{\bar\kappa}, \qquad (3\cdot5)$$

$$\partial_t K_{\bar\nu\bar\mu 0}{}^{\bar\kappa} = 2\partial_{[\bar\nu} K_{|0|\bar\mu]0}{}^{\bar\kappa}. \qquad (3\cdot6)$$

Since (3·5) and (3·6) can be transformed into

$$\epsilon^{\bar\rho\bar\nu\bar\mu}\partial_t K_{\bar\nu\bar\mu\bar\lambda}{}^{\bar\kappa} = 2\partial_{\bar\sigma}(\epsilon^{\bar\sigma\bar\rho\bar\mu} K_{0\bar\mu\bar\lambda}{}^{\bar\kappa}),$$
$$\epsilon^{\bar\rho\bar\nu\bar\mu}\partial_t K_{\bar\nu\bar\mu 0}{}^{\bar\kappa} = 2\partial_{\bar\sigma}(\epsilon^{\bar\sigma\bar\rho\bar\mu} K_{0\bar\mu 0}{}^{\bar\kappa}),$$

respectively, $K_{0\bar\mu\bar\lambda}{}^{\bar\kappa}$ can be interpreted as the flow of the spatial incompatibility, and $K_{0\bar\mu 0}{}^{\bar\kappa}$ as the flow of $K_{0[\bar\mu\bar\nu]}{}^{\bar\kappa}$.[1] Hence we have

Theorem 2. *The flow of the spatial incompatibility is represented by* $K_{0\bar\mu\bar\lambda}{}^{\bar\kappa}$ *for which the equation of continuity holds,*

$$\partial_t(\epsilon^{\bar\rho\bar\nu\bar\mu} K_{\bar\nu\bar\mu\bar\lambda}{}^{\bar\kappa}) = 2\partial_{\bar\sigma}(\epsilon^{\bar\sigma\bar\rho\bar\mu} K_{0\bar\mu\bar\lambda}{}^{\bar\kappa}).$$

Taking into account that the incompatibility changes as the dislocations move, from the distant parallelism criterion (i.e. the vanishing of the Riemann-Christoffel curvature tensor), we have

$$K_{\nu\mu\lambda\kappa} = 2\partial_{[\nu} S_{|\lambda\kappa|\mu]} + 2\partial_{[\lambda} S_{|\nu\mu|\kappa]}. \qquad (3\cdot7)^{2)}$$

Splitting this equation into the spatial and the mixed parts, we have

$$\left.\begin{aligned} K_{\bar\nu\bar\mu\bar\lambda\bar\kappa} &= 2\partial_{[\bar\nu} S_{|\bar\lambda\bar\kappa|\bar\mu]} + 2\partial_{[\bar\lambda} S_{|\bar\nu\bar\mu|\bar\kappa]}, \\ K_{0\bar\mu\bar\lambda\bar\kappa} &= \partial_t S_{\bar\lambda\bar\mu\bar\kappa} + 2\partial_{[\bar\lambda} S_{|0\bar\mu|\bar\kappa]}, \\ K_{\bar\nu 0\bar\lambda 0} &= 2\partial_t S_{0(\bar\lambda\bar\nu)}. \end{aligned}\right\} \qquad (3\cdot8)$$

3·2. Anelasticity tensor. The change of the plastic state is represented by the change of the images of the mapping in the natural state. Let a spatial vector $s^{\bar\lambda}$ be mapped to $s^{\bar a}$ in the natural state, and after a time dt let the same material vector be mapped to $s^{\bar a} + \Delta s^{\bar a}$. Since there is no distortion in the natural state, we can conclude that some plastic deformation occurs in the material constituting the vector $s^{\bar\lambda}$. The plastic change $\Delta s^{\bar a}$, or $\Delta s^{\bar\kappa} = B^{\bar\kappa}_{\bar a}\Delta s^{\bar a}$, can be expressed by the torsion tensor as

$$\Delta s^{\bar\kappa} = -\frac{1}{2} S_{0\bar\lambda}{}^{\bar\kappa} s^{\bar\lambda} dt. \qquad (3\cdot9)$$

Hence we see that the mixed components of the torsion tensor $S_{0\bar\lambda}{}^{\bar\kappa}$ represent the time rate of purely plastic deformation in agreement with §2.[3] The symmetric part $S_{0(\bar\lambda\bar\kappa)}$ denotes the plastic deformation and the antisymmetric part $S_{0[\bar\lambda\bar\kappa]}$ denotes the plastic rotation, each of which corresponds to the deformation and the rotation of the image in the natural state. Since the natural length of a material is measured by the length in its natural state, the length is changed by the plastic deformation (the elastic deformation does not affect it). The rate of the change will be obtained using the symmetric part $S_{0(\bar\lambda\bar\kappa)}$.

Eckart defines the anelasticity tensor $a_{\bar\lambda\bar\kappa}$ [9] by

$$\frac{D d\bar s^2}{Dt} = 2a_{\bar\lambda\bar\kappa} dx^{\bar\lambda} dx^{\bar\kappa}. \qquad (3\cdot10)$$

where

$$d\bar s^2 = g_{\bar\lambda\bar\kappa} dx^{\bar\kappa} dx^{\bar\lambda}.$$

Theorem 3. *The anelasticity tensor, by which the time rate of the change of the natural metric tensor is represented, is twice the symmetric part of the mixed component* $S_{0(\bar\lambda\bar\kappa)}$ *of the torsion tensor,*

$$a_{\bar\lambda\bar\kappa} = 2S_{0(\bar\lambda\bar\kappa)}. \qquad (3\cdot11)$$

Proof: By calculating

$$\begin{aligned} D d\bar s^2 &= \left(g_{\bar\lambda\bar\kappa} + \frac{\partial g_{\bar\lambda\bar\kappa}}{\partial t} dt\right) dx^{\bar\lambda}_{(t+dt)} dx^{\bar\kappa}_{(t+dt)} \\ &\quad - g_{\bar\lambda\bar\kappa} dx^{\bar\lambda} dx^{\bar\kappa} \\ &= \left\{\frac{\partial g_{\bar\lambda\bar\kappa}}{\partial t} + 2\partial_{(\bar\lambda} v_{\bar\kappa)} dx^{\bar\lambda} dx^{\bar\kappa}\right\} dt, \end{aligned}$$

where

$$dx^{\bar\lambda}_{(t+dt)} = dx^{\bar\lambda} + \frac{\partial v^{\bar\lambda}}{\partial x^{\bar\mu}} dx^{\bar\mu} dt,$$

and higher order terms are neglected, we find

$$a_{\bar\lambda\bar\kappa} = \frac{1}{2}\left\{\frac{\partial g_{\bar\lambda\bar\kappa}}{\partial t} + 2\partial_{(\bar\lambda} v_{\bar\kappa)}\right\}. \qquad (3\cdot12)$$

Hence

$$\begin{aligned} a_{\bar\lambda\bar\kappa} &= \partial_{(\bar\lambda} v_{\bar\kappa)} - \partial_t \varepsilon_{\bar\lambda\bar\kappa} \\ &= \partial_{(\bar\lambda} v_{\bar\kappa)} - \partial_t \beta_{(\bar\lambda\bar\kappa)} = 2S_{0(\bar\lambda\bar\kappa)}. \end{aligned}$$

The equation connecting $a_{\bar\lambda\bar\kappa}$ with the dislocation field is obtained from (2·9) :

1) Note that $K_{[\bar\nu\bar\mu 0]}{}^{\bar\kappa} = 0$ holds.

2) See the formula (4.22b) on p. 141 of [12].

3) It should be noted that elastic deformation does not affect the natural state.

$$4\partial_{[\nu}\partial_{|[\lambda}a_{\kappa]|\mu]}=2\partial_t(\partial_{[\nu}S_{|\lambda\kappa|\mu]}+\partial_{[\lambda}S_{|\nu\mu|]\kappa}). \quad (3\cdot13)$$

Using (3·8), we obtain the relation between the incompatibility and the anelasticity tensor:

$$K_{\nu 0\lambda 0}=\partial_t a_{\nu\lambda},$$
$$\partial_t K_{\nu\mu\lambda\kappa}=4\partial_{[\nu}\partial_{|[\lambda}a_{\kappa]|\mu]}. \quad (3\cdot14)$$

Hence, we have

Theorem 4. *The mixed components $K_{\nu 0\lambda 0}$ of the curvature tensor of the metric tensor $g_{\lambda\kappa}$ mean the rate of change of the anelasticity tensor.*

It is interesting to see that the rate of change of the spatial incompatibility of the strain is exactly the Riemann-Christoffel curvature tensor of the space having the anelasticity tensor as the Riemannian metric. In other words, we can define a Riemannian space which is the time derivative of the strain metric. It shows that the plastic deformation does not necessarily change the incompatibility $K_{\nu\mu\lambda}{}^{\kappa}$. Only such plastic deformations, whose anelasticity tensor is Riemannian (i.e. not Euclidean) or the deformation by which the image in the natural state is incompatibly changed, produces the spatial incompatibility.

It should also be noted that the antisymmetric part $S_{0[\lambda\kappa]}$ of the mixed part of the torsion tensor does not affect the natural metric. In other words, the plastic deformation such as denoted by $S_{0[\lambda\kappa]}$ induces neither the metrical imperfection nor the incompatibility.

4. Discussions

i) It seems to be but a step to extend this distant parallelism theory to a general non-Riemannian plasticity theory including moving curvature-like imperfections as well as torsion-like ones. We can proceed along the lines of D-IX [10]. Moreover, a four-dimensional non-Riemannian space can be imbedded in a Euclidean (or Minkowskian) space of higer dimensionality, in a manner similar to the way such as has been treated in the elegant theory of Kondo for three-dimensional case.[1] What rôle is played by the Euler-Schouten curvature tensor, which is pointed out to be more fundamental than the Riemann-Christoffel curvature tensor [5], is the next problem to be investigated

ii) The projective viewpoint has been suggested by Kondo in D-XII [14]. Both the curvature and the torsion tensor are combined into one geometrical concept, and the relation between the Riemann-Christoffel curvature tensor and the motion of dislocations shown in D-XI [7] will be clarified, if our point of view is extended to the projective one.

iii) It is not so difficult to obtain the stress and strain when a moving dislocation field is given. If an extended four-dimensional stress tensor is defined by

$$\sigma^{\lambda\kappa}=E^{\lambda\kappa\nu\mu}\varepsilon_{\nu\mu} \quad (4\cdot1)$$

where $E^{\lambda\kappa\nu\mu}$ is the ordinary elastic modulas tensor, $E^{\nu 0\mu 0}=\rho\delta^{\nu\mu}$, ρ is the density of the material, and the other independent components vanish, the wave equation is represented by

$$\partial_\lambda\sigma^{\lambda\kappa}=0. \quad (4\cdot2)$$

Since the curvature tensor calculated from the metric tensor $g_{\lambda\kappa}$ is, on account of vanishing of the Riemann-Christoffel curvature tensor, related to the torsion tensor by (3·7), we can obtain the stress and strain by solving this combined with (4·1), and (4·2).[2] Schaefer's stress function tensor will be extended to the four-dimensional case.

iv) From the macroscopic viewpoint, dislocations exist as dislocation lines in the three-dimensional space. They constitute a one-dimensional network in which the conservation law of Burgers vectors in analogy with Kirchhoff's law holds.

Since the trace of a dislocation line forms a surface in the four-dimensional space, it will be expected that the dislocations constitute a two-dimensional space. The conservation law of Burgers vectors will also hold by virtue of the non-divergent character of the torsion field. The career of a dislocation, e.g. the birth, growth, death, etc., will be represented by the two-dimensional dislocation network.

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1) See, e.g., [4], [5].

2) See [18].

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