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Theoretical Study of Oscillator Neurons in Recurrent Neural Networks

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***Abstract*— Neurons in a network can be both active or inactive. Given a subset of neurons in a network, is it possible for the subset of neurons to evolve to form an active oscillator by applying some external periodic stimulus? Furthermore, can these oscillator neurons be observable, that is, is it a stable oscillator? This paper explores such possibility, finding that an important property: any subset of neurons can be intermittently co-activated to form a stable oscillator by applying some external periodic input without any condition. Thus, the existing of intermittently active oscillator neurons is an essential property possessed by the networks. Moreover, this paper shows that, under some conditions, a subset of neurons can be fully co-activated to form a stable oscillator. Such neurons are called selectable oscillator neurons. Necessary and sufficient conditions are established for a subset of neurons to be selectable oscillator neurons in linear threshold recurrent neuron networks. It is proved that a subset of neurons forms selectable oscillator neurons if and only if the real part of each eigenvalue of the associated synaptic connection weight submatrix of the network is not larger than one. This simple condition makes the concept of selectable oscillator neurons tractable. The selectable oscillator neurons can be regarded as memories stored in the synaptic connections of networks, which enables to find a new perspective of memories in neural networks, different from the equilibrium-type attractors.**

***Index Terms*— Fully active oscillator neurons, intermittently active oscillator neurons, oscillators, recurrent neural networks, selectable oscillator neurons.**

I. Introduction

THERE have been many studies on the dynamics of recurrent neural networks (see [6], [23], [30]). They have single or multiple point attractors [1], [2], [15], [25], oscillations [7], [11], [18], [21], and chaotic dynamics [10] under constant inputs. Associative memory has also been studied by using such networks, where point attractors correspond to memorized patterns [3], [12], [13] under symmetric connections. In the asymmetric case, they can store sequences of patterns or cycles of patterns [3]. When external inputs are appropriately selected, they also show interesting phenomena, including selectable excitation patterns [8], [9], [22], [28]. The bump solutions of a neural field are also such an example [5].

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This paper studies selectable oscillation patterns in recurrent neural networks evoked by adequate external periodic stimuli.

Oscillations are ubiquitous in the brain [4], playing an important role not only in periodic behaviors but also in memories and integrating cortical activities of various areas. Furthermore, due to their property of feature binding, neural oscillators have been used to model perceptual processing in the brain [16], [26]. Can such a property also be possessed in artificial recurrent neural networks? Moreover, how can we describe such a property? [17], [19], [24], [29] provided clues in the regard. A recurrent neural network contains many neurons, each which can be active or inactive. Co-active neurons can form interesting dynamical patterns. In this paper, we ask. Do oscillator neurons exist universally in recurrent neural networks? That is, can the dynamics of any subset of neurons have the ability to evolve into co-activating toward a stable oscillator under adequate external periodic inputs? We demonstrate that any subset of neurons can *intermittently* co-activate into a stable oscillator by adequately selecting external inputs. Interestingly, no condition is required on the synaptic connections for the network to have such a property. This suggests that all neurons in a network have equal potential to act as oscillator neurons. As far as we know, the property of "intermittently active oscillator neurons" has not been studied in related work yet.

In this paper, we also explore under which conditions a given subset of neurons in the network can evolve into a *fully* active oscillator by applying adequate periodic outside stimulus. Furthermore, can these oscillator neurons be observable? That is, can they form a stable oscillator? We call such neurons "selectable oscillator neurons" and identify the necessary and sufficient conditions for a subset of neurons to be selectable oscillator neurons. Selectable oscillator neurons can be regarded as memories stored in the synaptic connections of the network, this is because that by applying some periodic outside stimulus to the network, some selectable oscillator neurons can be selected.

Different from traditionally regarding the equilibrium-type attractors as memories in recurrent neural networks, the concept of selectable oscillator neurons enables to find a new perspective of memories. The differences are obvious. First, the equilibrium-type attractors require that the external inputs of the networks must be constant. On the other hand, the selectable oscillator neurons require the external inputs be variable. From the mathematical view, the dynamics between these two types of networks have quite different properties. Second, the memory retrieving methods are different. The memory retrieving in equilibrium-type attractor's networks is via internal inputs, while in the networks with selectable oscillator neurons, the memory retrieving is via external inputs. From the view of network design, memory retrieving via external inputs is more flexible.

The rest of this paper is organized as follows. Preliminaries are given in Section II. The concepts and basic theories of the intermittently active oscillator neurons, and the selectable oscillator neurons are presented in Sections III and IV, respectively. Finally, conclusions are given in Section V.

II. Preliminaries

The model of recurrent neural networks studied in this paper is described as

$$\dot{x}(t) + x(t) = [Wx(t) + h(t)]^+, \quad t \geq 0$$

or in a components form

$$\dot{x}_i(t) + x_i(t) = \left[\sum_{j=1}^{n} w_{ij}x_j(t) + h_i(t)\right]^+, \quad t \geq 0$$

where $[s]^+ = \max\{s, 0\}$ is the linear threshold function, $x_i (i = 1, 2, \cdots, N)$ represents the activity of neuron i, $x(t) = (x_1(t), x_2(t), \cdots, x_N(t))^T \in \mathbb{R}^N$ denotes the state of the network at time t, $W = (w_{ij})_{N\times N}$ is the synaptic weight connection matrix of the network, and $h(t) \in \mathbb{R}^N$ denotes the external stimulus assumed to be a nonconstant continuous periodic function with period ω, i.e., $h(t + \omega) = h(t)$.

The output of the network is defined as

$$y(t) = [Wx(t) + h(t)]^+ \in \mathbb{R}^N$$

for $t \geq 0$. Each $y_i(t)$ is regarded as the output of neuron i at time t, so that clearly, $y_i(t) \geq 0$ for any $t \geq 0$. The activity of a neuron at time t is defined according to the value of $y_i(t)$.

Definition 1: If $y_i(t) > 0$, we say neuron i is active at time t. If $y_i(t) = 0$, we say neuron i is inactive at time t.

Next, we define the oscillator of the network.

Definition 2: An output $y^*(t)$ of the network is called an oscillator if $y^*(t + \omega) = y^*(t)$ for all $t \geq 0$.

In practice, only stable oscillators in a network can be observed, while unstable oscillators cannot be observed since any small disturbance could easily destroy them. An oscillator is stable if any output starting from points sufficiently close to the orbit stay close to the orbit all the time. The mathematical definition is given as follows.

Definition 3: The oscillator $y^*(t)$ is called stable, if for any $\epsilon > 0$, there exists a $\delta > 0$ such that $\|y(0) - y^*(0)\| \leq \delta$ implies that

$$\|y(t) - y^*(t)\| < \epsilon$$

for all $t \geq 0$. If an oscillator is not stable, it is called unstable.

It is well known that there are two viewpoints for inputs in recurrent neural networks. An external input h is fixed in one viewpoint and the initial vector is used as a network input. The other is that an initial vector $x(0)$ of the network is fixed and instead an external stimulus is used as the network input. In this paper, we take an external stimulus as the network input because this is in favor of network design. Fig. 1 gives an intuitive illustration.

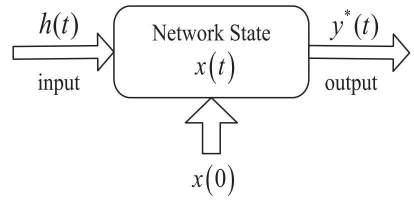

Fig. 1. Relationship between input and output of the network.

The two viewpoints for inputs imply two different methods of computing essentially. In fact, the network structure is unchanging when initial point $x(0)$ is used as computing input and external input h is fixed, while the network structure is changing by taking h as computing input since h is changing. It is well known from basic theory of differential equations that different structures of differential equations result in different dynamical behaviors.

Many research on dynamics of recurrent neural networks fix external input h and take initial point $x(0)$ as computing input, then the equilibrium-point attractors can be looked as memories stored in networks (see [14], [20], [27]). However, in this way, the attractors cannot encode external inputs. In order to encode more information of external input in the computing of networks and for easy of network design in practical applications, we take external inputs as computing inputs and present the concept of "selectable oscillator neurons" to indicate memories stored in networks. This mechanism is different from conventional equilibrium-type attractors.

Given a subset of neurons indexed by P, the dynamics (1) can be decomposed as

$$\begin{cases} \dot{x}_P(t) + x_P(t) = [W_P \cdot x_P(t) + W_{PZ} \cdot x_Z(t) + h_P(t)]^+ \\ \dot{x}_Z(t) + x_Z(t) = [W_{ZP} \cdot x_P(t) + W_Z \cdot x_Z(t) + h_Z(t)]^+ \end{cases}$$

where $P \subseteq \{1, 2, \cdots, N\}$ is a given index set and $Z = \{1, 2, \ldots, N\} \setminus P$, x_P and x_Z are subvectors of x constructed from x by removing the elements in x not indexed by P and Z, respectively, h_P, and h_Z are subvectors of h, W_P, and W_Z are principal submatrices of W constructed by removing from W all rows and columns not indexed by P and Z, respectively, W_{PZ} is a submatrix constructed from W by removing from W all rows not indexed by P and all columns not indexed by Z, and W_{ZP} is constructed in similar way. Accordingly, the output can be decomposed into $y_P(t)$ and $y_Z(t)$.

In recurrent neural networks, we are always interested in long time evolution of the output of each neuron. We are interested in knowing if a given subset of neurons can evolve to form some kind of active oscillator by adequate external stimulus.

III. Intermittently Active Oscillator Neurons

In this section, we address the problem of given any subset of neurons, does there exist any external periodic input to stimulate the network so that these neurons will evolve into stable active oscillators without any condition? If so, this would be a natural property existing in the network. The oscillator must be stable so that it can be observed, and it must also be active so that interesting patterns can be

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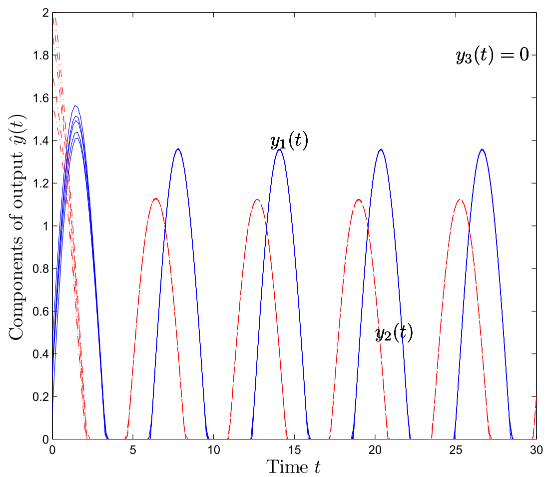


Fig. 2. Intermittently active oscillator neurons $\{1, 2\}$ in the network (1). The external periodic input is $h(t) = (\sin(t), \cos(t), -20 + \cos(t))^T$.

formed. To illustrate this point, we consider a network in three dimensions and randomly generate the connection matrix W as

$$W = \begin{bmatrix} 0.1220 & 0.3317 & 0.1217 \\ 0.2684 & 0.1522 & 0.8842 \\ 0.2578 & 0.3480 & 0.0943 \end{bmatrix}.$$

Given a subset of neurons with index set $P = \{1, 2\}$, we choose the external periodic input as

$$h(t) = [\sin(t), \cos(t), -20 + \cos(t)]^T.$$

Fig. 2 clearly shows that network dynamics evolve into a stable oscillator, and those neurons indexed by P evolve into *intermittently active oscillator neurons* while the rest of neurons evolve into 0. We can easily check that any subset of neurons are all *intermittently active oscillator neurons* by adequate external inputs. It can be observed in Fig. 2 that the *intermittent activity* of neurons exhibit an interesting property that they are periodically active in some intervals and inactive in others.

Next, we give a formal definition to this phenomenon. First, we provide some notations. A vector $x \in \mathbb{R}^n$ is said to be nonnegative, denoted by $x \geq 0$, if each component of x is nonnegative. A vector $x \in \mathbb{R}^n$ is said to be positive, denoted by $x > 0$, if each component of x is positive.

Definition 4: When there exists periodic input $h(t)$ such that a stable oscillatory solution $x^*(t)$ exists that satisfies $y_P^*(t) > 0$ in some intervals and $y_P^*(t) = 0$ in the other intervals periodically, and $y_Z^*(t) = 0$, P is called a set of intermittently active oscillator neurons.

Theorem 1: For any set of neurons with index set P, there exists a periodic input $h(t)$ for which a stable Intermittently Active Oscillator $x^*(t)$ exists.

Theorem 1 explores an important property that any subset of neurons can be intermittently co-activated to form a stable oscillator by applying some external periodic input without any condition. It shows that the existing of intermittently active oscillator neurons is an essential property possessed by the networks.

Next, we are going to prove this theorem. We will try our best to put the proof in an easy understand way.

It is clear that, when $h_Z(t)$ is chosen to be negative enough, the activity of neurons in Z is suppressed so that we may put $y_Z^*(t) = 0$. Then we only need to consider the subsystem

$$\dot{x}_P(t) = -x_P(t) + [Wx_P(t) + h_P(t)]^+, \quad t \geq 0.$$

For convenience, we omit suffix P in the proof, considering only neurons in P, and denote $A = (I - W)_P$. We further put $\omega = 1$ without loss of generality by choosing an adequate time scale. Before describing a formal proof, we consider a linear differential equation

$$\dot{x}(t) = \begin{cases} -Ax(t) + a, & 0 \leq t < \tau \\ -x(t), & \tau \leq t \leq 1 \end{cases}$$

where $a > 0$ is constant. This coincides with the original (1) with the input

$$h(t) = \begin{cases} a, & 0 \leq t < \tau \\ -b, & \tau \leq t \leq 1 \end{cases}$$

where $a > 0$ is large enough to hold $Wx(t) + a > 0$ and $-b < 0$ is small enough to hold $Wx(t) - b < 0$. Clearly, it holds that

$$y(t) \begin{cases} > 0, & 0 \leq t < \tau \\ = 0, & \tau \leq t \leq 1. \end{cases}$$

The solution of (1) is

$$x(t) = \begin{cases} e^{-At}x_0 + (I - e^{-At})A^{-1}a, & 0 \leq t \leq \tau \\ e^{-(t-\tau)}x(\tau), & \tau \leq t \leq 1 \end{cases}$$

where $x_0 = x(0)$. We define two matrices

$$X = e^{-(1-\tau)}e^{-A\tau}$$
$$Y = e^{-(1-\tau)}(I - e^{-At})A^{-1}$$

which depend on τ. We then have

$$x_1 = Xx_0 + Ya, \quad \text{where } x_1 = x(1).$$

We search for a solution $x^*(t)$ that satisfies

$$x^* = x^*(0) = x^*(1).$$

From (1), its initial (and final) state is given by

$$x^* = (I - X)^{-1}Ya$$

and hence

$$x^*(t) = e^{-At}x^* + (I - e^{-At})A^{-1}a = G(t, \tau)a$$

for $0 \leq t < \tau$, where

$$G(t, \tau) = e^{-At}(I - X)^{-1}Y + (I - e^{-At})A^{-1}.$$

The proof of Theorem 1 then consists of the following four lemmas.

Lemma 1: There exists a constant $0 < \tau < 1$ and two vectors $a > 0$ and $-b < 0$ such that

$$Wx^*(t) + a > 0, \quad 0 \leq t \leq \tau$$
$$Wx^*(t) - b < 0, \quad \tau \leq t \leq 1.$$

Proof: Since $x^*(t)$ is bounded, the existence of b is obvious. We evaluate $G(t,\tau)$ for sufficiently small τ and $t < \tau$. Since Y and $I - e^{-At}$ are of order τ while e^{-At} and $(I-X)^{-1}$ are bounded, we have $G(t,\tau) = O(\tau)$. From (1) there exists $a > 0$, at least for sufficiently small τ. This proves the lemma. ■

By using the property of continuity, the result below follows directly from Lemma 1.

Lemma 2: There exists a small neighborhood N_{x^*} of x^* such that for any $x_0 \in N_{x^*}$, it holds that

$$\begin{cases} Wx(t) + a > 0, & 0 \le t \le \tau \\ Wx(t) - b < 0, & \tau \le t \le 1 \end{cases}$$

where $x(t)$ is the trajectory starting from x_0.

Based on τ, a, and b in Lemma 1, we now choose an input of network (1) with a period of 1 (the case with an arbitrary period ω is also true) as

$$h(t) = \begin{cases} a, & n \le t < n+\tau \\ -b, & n+\tau \le t < n+1 \end{cases}$$

where $n = 0, 1, 2, \ldots$. Since $x^*(0) = x^*(1)$ and $h(t)$ are periodic, the following lemma extends the solution to $t \in [n, n+1], n = 0, 1, 2, \ldots$, thus proving that there exists a periodic solution $x^*(t)$ of network (1) with this input.

Lemma 3: With the input defined by (1), the solution $x^*(t)$ is extended over all $t \ge 0$ with period 1 by $x^*(t+1) = x^*(t)$.

There is only one step left to complete the proof of Theorem 1: the stability of period solution $x^*(t)$ defined by (1). The following lemma ends it.

Lemma 4: Given any $x(0) \in N_{x^*}$, there exists $\eta > 0$ such that

$$\|x(n) - x^*(n)\| \le e^{-\eta n}\|x(0) - x^*(0)\|$$

for all $n = 0, 1, 2, \ldots$

Proof: Let $x(t)$ be the solution starting at x_0. Then, it becomes $x_1 = Xx_0 + Ya$.

The mapping $x_0 \to x_1$ is the *Poincaré map* of the periodic orbit $x^*(t)$. Therefore, the initial deviation $\delta x_0 \triangleq x_0 - x^*$ changes to $\delta x_1 \triangleq x_1 - x^*$ as $\delta x_1 = X\delta x_0$.

Let λ_i be the eigenvalues of A with corresponding eigenvectors v_i such that $Av_i = \lambda_i v_i$. Then, we have

$$Xv_i = e^{-(1-\tau)}e^{-\lambda_i \tau}v_i$$

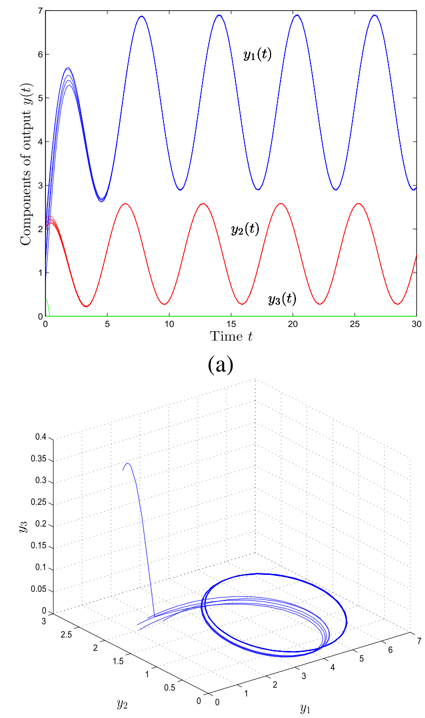
showing that the eigenvalues of X are $e^{-(1-\tau)}e^{-\lambda_i\tau}$. Let us take $\lambda = \max_i \left\{e^{-(1-\tau)}e^{-\lambda_i\tau}\right\}$, then

$$\|\delta x_1\| \le \lambda\|\delta x_0\|.$$

By choosing a sufficiently small τ, we have $\lambda < 1$. Hence, by letting $-\eta = \log\lambda$, $\eta > 0$, we have

$$\|\delta x_1\| \le e^{-\eta}\|\delta x_0\|.$$

This repeats, proving Lemma 4, as well as Theorem 1. ■



(b)

Fig. 3. Set of selectable oscillator neurons $\{1, 2\}$ driven by $h(t) = (2 + \sin(t), 1 + \cos(t), \sin(t))^T$ in the network (1). (a) Components of $y^*(t)$ with the property of $y_i^*(t+\omega) = y_i^*(t) > 0 (i = 1, 2)$ and $y_3^*(t) \equiv 0$ for all $t \ge 0$ and $\omega = 2k\pi (k = 0, 1, 2, \ldots)$. (b) Stable oscillator $y^*(t)$ located in the coordinate plane $y_3 = 0$.

IV. Selectable Oscillator Neurons

In this section, we study the problem of under what condition can a given subset of neurons evolve to co-activate *Fully* into a stable oscillator by applying some external periodic inputs. Such neurons are called *Selectable Oscillator Neurons*. We will identify the necessary and sufficient conditions for this.

To further illustrate the concept of selectable oscillator neurons, let us first consider an network for example

$$\begin{cases} \dot{x}_1(t) + x_1(t) = [0.3x_1(t) + x_1(t) - 2x_3(t) + h_1(t)]^+ \\ \dot{x}_2(t) + x_2(t) = [0.3x_2(t) + h_2(t)]^+ \\ \dot{x}_3(t) + x_3(t) = [-3x_1(t) + x_2(t) + 0.3x_3(t) + h_3(t)]^+ \end{cases}$$

for $t \ge 0$.

The neurons with index set $P_1 = \{1, 2\}$ are selectable oscillator neurons. In fact, by choosing

$$h(t) = (2 + \sin(t), 1 + \cos(t), \sin(t))^T$$

it shows in Fig. 3 that the output of the network evolves to a stable oscillator $y^*(t)$ located on the coordinate plane $y_3 = 0$, i.e., any trajectory starting from a point sufficiently close to the oscillator will stay close to the oscillator forever.

Fig. 4 shows another selectable oscillator set of neurons with index set $P_2 = \{2, 3\}$ driven by external input

$$h(t) = (0.5\sin(t), 2 + \sin(t), 3\cos(t))^T$$

which is located in the coordinate plane $y_1 = 0$.

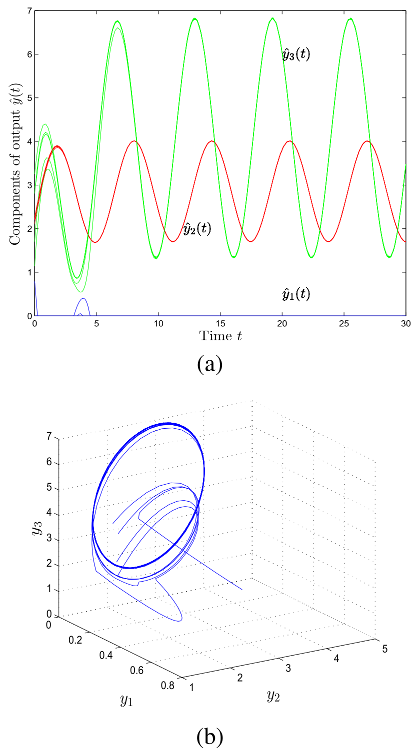


Fig. 4. Set of selectable oscillator neurons $\{2, 3\}$ driven by $h(t) = (0.5\sin(t), 2+\sin(t), 3\cos(t))^T$ in the network (1). (a) Components of $\hat{y}^*(t)$ with the property of $\hat{y}_1^*(t) \equiv 0$ and $\hat{y}_i^*(t+\omega) = \hat{y}_i^*(t) > 0 (i = 2, 3)$ for all $t \geq 0$ and $\omega = 2k\pi\,(k = 0, 1, 2, \ldots)$. (b) Stable oscillator $\hat{y}^*(t)$ located in the coordinate plane $y_1 = 0$.

However, it should be noted that there exists an unselectable set of oscillator neurons in this network, i.e., the set of neurons $\{1, 3\}$ since there does not exist any stable oscillator y^* with $y_i^*(t+\omega) = y_i^*(t) > 0 (i = 1, 3)$ no mater which external input $h(t)$ is chosen.

Moreover, it is interesting that, in the above scheme, an external stimulus is taken as an input of the network for computing. This is different from traditional computing in recurrent neural networks that an initial value is taken as an input for computing, but not external stimulus.

Definition 5: A set of neurons with index set P is called a set of selectable oscillator neurons if there exists an external periodic input $h(t+\omega) = h(t)$ such that the network (1) possesses a stable oscillator $y^*(t)$ with properties

$$\begin{cases} y_i^*(t+\omega) = y_i^*(t) > 0 (i \in P) \\ y_j^*(t) \equiv 0 (j \notin P) \end{cases}$$

for all $t \geq 0$. In other words, neurons in P will co-activate into a stable oscillator.

Theorem 2: A set of neurons with index set P is a set of selectable oscillator neurons, if and only if the real part of each eigenvalue of the submatrix W_P is not larger than 1.

This condition is quite simple and tractable, hence, it gives a simple way to design an oscillator network.

Proof: The proof requires necessary and sufficient parts. For the necessary part, suppose that a set of neurons with index set P are selectable oscillator neurons. We will prove that the real part of each eigenvalue of W_P is not larger than 1. By Definition 5, there exists an external periodic input $h(t+\omega) = h(t)$ such that the network (1) has a stable oscillator $y^*(t)$ with the property

$$\begin{cases} y_P^*(t+\omega) = y_P^*(t) > 0 \\ y_Z^*(t) \equiv 0 \end{cases}$$

for all $t \geq 0$. From (1), it must hold that

$$\begin{cases} \dot{x}_P^*(t) = (W_P - I)\cdot x_P^*(t) + W_{PZ}\cdot x_Z^*(t) + h_P(t) \\ \dot{x}_Z^*(t) + x_Z^*(t) = 0 \end{cases}$$

for $t \geq 0$, where I is the unit matrix. Suppose there exists a eigenvalue λ such that $Re(\lambda) > 0$ for the matrix $(W_P - I)$. Let ξ be the associated eigenvector of λ. Then, it follows that:

$$\frac{d\xi^T x_P^*(t)}{dt} = \lambda\cdot\xi^T x_P^*(t) + \xi^T W_{PZ} x_Z^*(0)e^{-t} + \xi^T h_P(t)$$

for $t \geq 0$. Thus

$$\begin{aligned} \xi^T x_P^*(t) &= e^{\lambda t}\times\left[\frac{\xi^T W_{PZ} x_Z^*(0)}{\lambda+1}\left(1 - e^{-(\lambda+1)t}\right)\right. \\ &\qquad\left. + \xi^T x_P^*(0) + \int_0^t e^{-\lambda s}\cdot\xi^T h_P(s)ds\right] \\ &\rightarrow \infty \end{aligned}$$

as $t \rightarrow +\infty$, since

$$\int_0^t e^{-\lambda s}\cdot\xi^T h_P(s)ds$$

is bounded as h_P is bounded. This contradicts the fact that x_P^* is bounded, which proves that each eigenvalue of W_P must be not larger than 1. This completes the proof of the necessary part.

To prove the sufficient part, suppose that the real part of each eigenvalue of W_P is not larger than 1. We will show that the neurons in P must be selectable oscillator ones.

Choosing an external periodic stimulus as

$$\begin{cases} h_P(t) = (3 + \sin t + \cos t)\cdot 1_P - (3 + \sin t)W_P\cdot 1_P \\ h_Z(t) = -1_Z - (3 + \sin t)W_{ZP}\cdot 1_Z \end{cases}$$

for $t \geq 0$, where 1_P and 1_Z are two vectors with components being 1, it is easy to see that the network has an oscillator defined by

$$\begin{cases} y_P^*(t) = [W_P\cdot x_P^*(t) + h_P(t)]^+ = (3 + \sin t + \cos t)\cdot 1_P \\ y_Z^*(t) = [W_{ZP}\cdot x_P^*(t) + h_Z(t)]^+ \equiv 0 \end{cases}$$

for all $t \geq 0$.

To complete the proof, we only need to show that the oscillator $y^*(t)$ is stable. Since

$$y^*(t) = [Wx^*(t) + h(t)]^+$$

it is sufficient to prove that $x^*(t)$ is stable.

Clearly, we have

$$\begin{cases} W_P\cdot x_P^*(t) + h_P(t) = 3 + \sin t + \cos t \geq 1_P > 0 \\ W_{ZP}\cdot x_P^*(t) + h_Z(t) = -1_Z < 0 \end{cases}$$

for $t \geq 0$. The dynamics starting at $x_P(0)$ and $x_Z(0)$ can be rewritten as

$$\begin{cases} \dot{x}_P(t) + x_P(t) = \big[W_P \cdot \big(x_P(t) - x_P^*(t)\big) + W_{PZ} \cdot x_Z(t) \\ \qquad\qquad + (3 + \sin t + \cos t) \cdot 1_P\big]^+ \\ \dot{x}_Z(t) + x_Z(t) = \big[-1_Z + W_{ZP} \cdot \big(x_P(t) - x_P^*(t)\big) \\ \qquad\qquad + W_Z \cdot x_Z(t)\big]^+ \end{cases}$$

for all $t \geq 0$.

Given any ϵ such that

$$0 < \epsilon \leq \min\left\{\frac{1}{\|W_P\|+1}, \frac{1}{\|W_{PZ}\|+1}, \frac{1}{\|W_{ZP}\|+1}, \frac{1}{\|W_Z\|+1}\right\}$$

define a constant

$$\delta = \frac{\epsilon}{\max\{\sqrt{2}1 + 4\|W_{PZ}\|\}}.$$

To prove that x^* is stable, it is sufficient to prove

$$\|x_P(0) - x_P^*(0)\| + \|x_Z(0)\| < \delta$$

implies that

$$\|x_P(t) - x_P^*(t)\| + \|x_Z(t)\| < \epsilon$$

for all $t \geq 0$. If this is not true, there must exist a time $t_1 > 0$ such that

$$\begin{cases} \|x_P(t) - x_P^*(t)\| + \|x_Z(t)\| < \epsilon, \quad 0 \leq t < t_1 \\ \|x_P(t_1) - x_P^*(t_1)\| + \|x_Z(t_1)\| = \epsilon. \end{cases}$$

Then, for $i \in P$, it holds for $0 \leq t \leq t_1$ that

$$\begin{aligned} &\sum_{j\in P} w_{ij} \cdot \big(x_j(t) - x_j^*(t)\big) + \sum_{j\in Z} w_{ij} \cdot x_j(t) \\ &\quad + \left(2 + \frac{2\pi}{\omega} + \sin\frac{2\pi}{\omega}t + \frac{2\pi}{\omega}\cos\frac{2\pi}{\omega}t\right) \\ &\geq 1 - \|W_P\| \cdot \big\|x_P(t) - x_P^*(t)\big\| - \|W_{PZ}\| \cdot \|x_Z(t)\| \\ &\geq 1 - \max\{\|W_P\|, \|W_{PZ}\|\} \cdot \epsilon \\ &> 0 \end{aligned}$$

and for $i \in Z$, it holds for $0 \leq t \leq t_1$ that

$$\begin{aligned} &-1 + \sum_{j\in P} w_{ij} \cdot (x_j(t) - x_j^*(t)) + \sum_{j\in Z} w_{ij} \cdot x_j(t) \\ &\leq -1 + \|W_{ZP}\| \cdot \big\|x_P(t) - x_P^*(t)\big\| + \|W_Z\| \cdot \|x_Z(t)\| \\ &\leq -1 + \max\{\|W_{ZP}\|, \|W_Z\|\} \cdot \epsilon \\ &< 0. \end{aligned}$$

Thus, for $0 \leq t \leq t_1$, it follows from (1) that:

$$\frac{d\big[x_P(t) - x_P^*(t)\big]}{dt} = (W_P - I)\big[x_P(t) - x_P^*(t)\big] + W_{PZ}x_Z(t)$$

and

$$\dot{x}_Z(t) = -x_Z(t).$$

Clearly

$$x_Z(t) = x_Z(0)e^{-t}(0 \leq t \leq t_1).$$

Moreover

$$\begin{aligned} &\frac{d\big[\|x_P(t) - x_P^*(t)\|^2\big]}{dt} \\ &= -2\big[x_P(t) - x_P^*(t)\big]^T (I - W_P) \cdot \big[x_P(t) - x_P^*(t)\big] \\ &\quad + 2\big[x_P(t) - x_P^*(t)\big]^T W_{PZ} \cdot x_Z(t) \\ &\leq 2\big[x_P(t) - x_P^*(t)\big]^T W_{PZ} \cdot x_Z(t) \\ &\leq 2\big\|x_P(t) - x_P^*(t)\big\| \cdot \|W_{PZ}\| \cdot \|x_Z(t)\| \end{aligned}$$

for $0 \leq t \leq t_1$. Then

$$\begin{aligned} &\big\|x_P(t) - x_P^*(t)\big\|^2 \\ &\leq \big\|x_P(0) - x_P^*(0)\big\|^2 \\ &\quad + 2\int_0^t \big\|x_P(s) - x_P^*(s)\big\| \cdot \|W_{PZ}\| \cdot \|x_Z(0)\| \cdot e^{-s} ds \\ &\leq \big\|x_P(0) - x_P^*(0)\big\|^2 \\ &\quad + 2 \cdot \|W_{PZ}\| \cdot \|x_Z(0)\| \cdot \sup_{0\leq s\leq t_1} \big\|x_P(s) - x_P^*(s)\big\| \\ &\leq \big\|x_P(0) - x_P^*(0)\big\|^2 + 8 \cdot \|W_{PZ}\|^2 \cdot \|x_Z(0)\|^2 \\ &\quad + \frac{1}{2}\sup_{0\leq s\leq t_1} \big\|x_P(s) - x_P^*(s)\big\|^2 \end{aligned}$$

for $0 \leq t \leq t_1$. It follows that:

$$\big\|x_P(t) - x_P^*(t)\big\| \leq \sqrt{2}\big\|x_P(0) - x_P^*(0)\big\| + 4\|W_{PZ}\| \cdot \|x_Z(0)\|$$

for $0 \leq t \leq t_1$. Then

$$\begin{aligned} &\big\|x_P(t_1) - x_P^*(t_1)\big\| + \|x_Z(t_1)\| \\ &\leq \sqrt{2}\big\|x_P(0) - x_P^*(0)\big\| + (1 + 4\|W_{PZ}\|) \cdot \|x_Z(0)\| \\ &\leq \max\{\sqrt{2}, 1 + 4\|W_{PZ}\|\} \\ &\quad \times \big(\big\|x_P(0) - x_P^*(0)\big\| + \|x_Z(0)\|\big) \\ &< \epsilon. \end{aligned}$$

This leads to a contradiction, showing that $x^*(t)$ is stable, and thus $y^*(t)$ is stable. This completes the proof of the sufficient part, as well as the theorem. ■

V. Conclusion

The concepts of intermittently oscillator neurons and selectable oscillator neurons are proposed for a class of recurrent neural networks. The former explores a universal property of the network that the dynamics of any subset of neurons can co-activate intermittently at a stable oscillator by applying adequate external periodic inputs, while the latter suggests an alternative viewpoint to memory storage in recurrent neural networks that is different from the conventional equilibrium-type attractors. The most interesting result is that some neurons can be selectable oscillator neurons while others cannot. Suppose some memories are stored as selectable neurons. Then such memories can be reliably retrieved by applying a periodic stimulus from outside. This raises a question of how we can store the memories as selectable oscillator neurons. This paper provides a clear and tractable answer: the selectable oscillator neurons can be stored in the network simply by ensuring that the real part of each eigenvalue of the associated synaptic connection weight submatrix is not larger than 1. It can be expected that interesting applications could be developed on the basis of this theoretical study of oscillator neurons. This will be studied in the future.

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