証正用

F-VI

Topological Foundations of Kron's Tearing of Electric Networks

By

Shun-ichi AMARI

P 322—350

Reprinted from RAAG MEMOIRS OF THE UNIFYING STUDY OF BASIC PROBLEMS IN ENGINEERING AND PHYSICAL SCIENCES BY MEANS OF GEOMETRY
Vol. III, Division F, pp. 322~350, 1962

RAAG MEMOIRS
Vol. 3, 1962

F - VI

Topological Foundations of Kron's Tearing of Electric Networks

By
Shun-ichi AMARI*

[Received February 1961]

PREFACE

IT was in No. 21 of the RAAG Research Notes, Third Series, [1] that an idea for clarifying the intrinsic topological structure of diakoptics [2] and codiakoptics [3] in application of the theory of the topolopical dissection (see e. g. [4]) was first proposed and analyzed. Then, the information-theoretical foundation of the efficacy of diakoptics was investigated [5] based upon the topological foundation [1]. To these were added the diakoptical eigenvalue analysis [6] in the author's Master's Thesis [7] presented to the University of Tokyo in February 1960 as a summary of his investigations over two years (1958~60) in postgraduate courses under the guidance of Professor Kazuo Kondo at the Division of Research in Mathematical and Physical Sciences of the Graduate School. This paper is mainly based upon Part I of the Thesis, but an improvement has been made by adding a new section. The next paper F-VII [8] is based on Part II of the Thesis.

The author hopes that he has been able to propose some new ideas in this paper in connexion with topological analysis in the field of engineering as well as a contribution to the theory of diakoptics and codiakoptics by clarifying their essential or internal structure and extending the method.

The author is very happy to express his hearty gratitude to Professor Kondo for kind guidance and inspiring suggestions and to his colleagues, especially Dr. M. Iri, for discussions and many suggestions. The guidance and the suggestions of Professor Kondo used to appear very fantasic at first glance, but the author has found them to be essentially penetrating. Dr. Iri's discussions, covering detailed and important propositions, many of which were proposed by him, gave the author much material which has been included in this paper. But for these helps and the stimulating experiences during the later half of the four years of my life in Hongo Campus, this paper would have appeared in a more incomplete form.

INTRODUCTION

AS science and engineering have been developed astonishingly in recent years, we now have many more opportunities than before to treat large-scale systems in engineering analysis. But it is not an easy task to handle a large-scale system as a whole. Therefore, it is more convenient to tear, if possible, the original system first into several subsystems and then interconnect the solutions of the subsystems into the solution of the whole system. This is the idea first proposed and developed by Gabriel Kron into a practical method of solving large-scale system by tearing. A new word "Diakoptics" has been introduced for this method by the initiator himself.

There is no doubt that diakoptics gives powerful practical means of engineering analysis. It has already been treated by many investigators (e. g. [2], [3], [9], [10], [11], [12]

* University of Tokyo, Tokyo, Japan.

[13]), and its validity has been widely recognized. R. Onodera proposed the method of codiakoptics [3], [4]. K. Kondo extended the point of view to the generalized diakoptics [15]. Based on his idea, M. Iri and T. Sunaga discussed the efficacy of diakoptics [16]. Some of the topological features of Kron's diakoptical analysis has been discussed by J. P. Roth in [11] and [17]. R. R. Sabroff proposed some new concepts on diakoptics [13].

However, Roth's exposition [17] and the presentations in some of the papers (e. g. [2]) by Kron himself give us an impression of having avoided the expression we would prefer, as far as a formal statement of the basic philosophy is concerned. Sabroff's treatment [13], in which the equivalent circuits and the orthogonal network method [18] are combined, seems most nearly to approach our method. But his paper as well as those mentioned above (also W. A. Blackwell and H. H. Kesvan's paper [19] as well as J. P. Char's method of orthogonal networks [20]) seems, so to speak, to rely so much upon the physical images of the equivalent circuits of the torn subnetworks that they have a tendency to miss the mathematical intrinsic structure of diakoptics. Therefore, we would not say that the theoretical foundations have satisfactorily been given for the methodology of diakoptical tearing.

There are two features of approach to the foundations of diakoptics and codiakoptics. One is to investigate the topological features, for the most important relations between the entire system and the subsystems are their topological connexion relations. It is said that topological invariance is not preserved in diakoptical analysis [15]. But just at this very point, topology reveals its power. The efficacy of the diakoptical, as well as the codiakoptical, methods consists in utilizing the informations on the topological characteristics of the networks or manifolds to be treated by tearing the original system. Therefore, its theoretical foundations must be approached from the point of view of topology.

The other feature is to clarify the concept of the informations on networks. The graph of a network has something more than that which is translated into the equation derived from the graph. The "something" can be understood to be the information contained in a graph. Such considerations will be studied in F-VII [8] along the lines of F-IV [16]. The most important thing is to connect the above two features, i. e. the topological and information-theoretical viewpoints, into one. In this paper, we shall investigate the topological features of diakoptics. A short outline of the contents is as follows.

In the first section, a new and generalized approach to diakoptics, which has been derived by the topological considerations of later sections, will be shown as the introductory illustration of the practical importance of our theory. In sections 2, 3, 4, and 5, we shall try to give the topological foundations of diakoptics and codiakoptics of networks using the method of dissection of combinatorial topology (see [4], [21], [22] for the theory of topology). The fundamental equations of network diakoptics and codiakoptics are established (in §3, in more generalized form in §4), and the diakoptics and codiakoptics are proved to be essentially the same. Our method extends the ordinary one in regard to two points as follows. One is that the so-called "cut-branches" are extended to more generalized "cut-network $\overset{0}{X}$", and the other is that our method is applicable when mutual couplings exist between the branches of the subnetworks $\overset{1}{X}$ and those of the cut-network $\overset{0}{X}$. In §6 the intrinsic structure of diakoptical eigenvalue problems is clarified by a normal form of representation. Using this form, not only is Kron's method [2] refined, but also a new iterative method, which terminates with a finite number of iterations giving the exact solution, is proposed. In the last section (§7) the topological characteristics of subsystems, i. e. subnetwork $\overset{1}{X}$ and cut-network $\overset{0}{X}$, are studied. The number of unknown variables defined in connexion with diakoptics is discussed. It will be shown that the number of necessary variables may be less in the diakoptical analysis than in either the mesh or the node-pair analysis.

Lastly, the papers in [23], and especially [24] and [25] may be mentioned as indispensable references so far as the topological theory and terminology of networks used in this paper are concerned.

1. Introductory Illustration of a New Approach to Diakoptics and Codiakoptics

Kron's method of tearing—called "Diakoptics" by Kron himself—has mostly been illustrated with the help of the physical image of the equivalent circuit. This is, as a matter of course, a very easy method of illustration. But it seems that not all of the intrinsic structure of diakoptics could be revealed by relying too much upon the physical image. By investigating the diakoptical structure with the help of the topological theory of dissection, a quite new approach to diakoptics is made, by which the method is extended in several important respects. Moreover, diakoptics and its dual, codiakoptics [3], become unified, and either can be derived as a special case of our method.

In the first section, our method and the fundamental equation will be explained in a practical terminology but not rigorously. Its theoretical derivation as well as the topological foundation will be given in the subsequent sections.

1·1. Diakoptical coordinates of an electrical network. There have been known for a long time two methods of analyzing an electrical network. One is the mesh method in which independent mesh currents i^p ($p=1, 2, ..., k$: k is the number of the independent meshes) are chosen as unknown variables. The equation to be satisfied by the i^p's is

$$z_{qp}i^p = e_q, \tag{1·1}$$

where (z_{qp}) is the mesh-impedance matrix, e_q is the impressed e. m. f. in the q-th mesh, and the Einstein summation convention is assumed. Let us call the set (i^p) of the variables i^p the *mesh coordinates* of the network, since the excited state of the network is completely specified by this set.

The other method is the node-pair method in which independent node voltages E_a ($a=1, 2, ..., m$: m is the number of independent nodes, or $m+1$ is that of all the nodes in a connected network) are independent variables. The equation to be satisfied by them is

$$y^{ab}E_b = I^a, \tag{1·2}$$

where (y^{ab}) is the node-admittance matrix and I^a is the impressed node current at the a-th node. The actual excited state can also be specified by the set (E_b) of the variables E_b. Hence we call them the *node coordinates* of the network.[1)]

The above two coordinates are related to each other by Ohm's law. Let (D^a_λ) ($a=1, ..., m$; $\lambda=1, ..., n$: n is the number of branches) be the incidence matrix between nodes and branches, i. e. $D^a_\lambda=+1(-1)$, if the λ-th branch starts from (or ends with) the a-th node, and $D^a_\lambda=0$, otherwise. Then the branch voltage drop E_λ across the λ-th branch is

$$E_\lambda - e_\lambda = D^a_\lambda E_a, \tag{1·3}$$

where e_λ is the impressed electromotive force in the λ-th branch. The node-admittance matrix (y^{ab}) is, as is well-known, represented by

$$y^{ab} = D^a_\kappa y^{\kappa\lambda} D^b_\lambda, \tag{1·4}$$

where $(y^{\kappa\lambda})$ is the branch-admittance matrix.

In the dual manner, let (R^κ_p) be the incidence matrix between meshes and branches, i. e. $R^\kappa_p=+1(-1)$ if the κ-th branch is a constituent element of the p-th mesh with positive (or negative) orientation and $R^\kappa_p=0$, otherwise. Then the mesh-impedance matrix (z_{qp}) is

$$z_{qp} = R^\kappa_q z_{\kappa\lambda} R^\lambda_p, \tag{1·5}$$

where $(z_{\kappa\lambda})$ is the branch-impedance matrix. The branch current i^κ passing through the κ-th branch is given by

$$i^\kappa - I^\kappa = R^\kappa_p i^p, \tag{1·6}$$

where I^κ is the impressed current in the κ-th branch.

Ohm's law is expressed as

$$i^\kappa = y^{\kappa\lambda}E_\lambda \text{ or } E_\lambda = z_{\lambda\kappa}i^\kappa. \tag{1·7}$$

1) The orthogonal network method proposed by Kron [18] uses both the i^p's and E_a's as independent variables. Hence that set (i^p, E_a) may be called the *orthogonal coordinates* of the network. But in practical analysis, either the i^p's or the E_a's are eliminated according as to whether the impressed quantities e_q's or I^b's are converted to the equivalent node current sources or mesh voltage sources, and the coordinates will reduce to the mesh or node ones, respectively.

Then the two coordinates (i^p) and (E_a) are mutually related by

$$I^\kappa + R^\kappa_p i^p = y^{\kappa\lambda}(D^q_\lambda E_a + e_\lambda)$$

or

$$e_\lambda + D^q_\lambda E_a = z_{\lambda\kappa}(R^\kappa_p i^p + I^\kappa). \qquad (1\cdot8)$$

Let us define the diakoptical coordinates of the network which consist of mesh currents $i^p_0 (p=1,\ldots,k_0 \leqq k)$ on some part of the network, and of node voltages $E_a (a=1,\ldots,m_1 \leqq m)$ on the other part. To obtain a set of the non-redundant variables (i^p_0, E_a)[1], we may well choose the variables as follows.

First partition the branches into two parts (see Fig. 1 where the branches of the first part are denoted by the dotted lines, and those of the other parts by the full lines). Take up

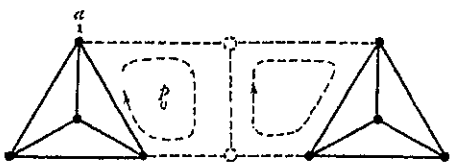


FIG. 1

all the independent meshes denoted by the dotted circles which have at least a branch belonging to the first part as its constituent elements, and choose the mesh currents i^p_0 as the variables. Next, delete the first part from the network and take up all the independent nodes, denoted by the black points which are incident at least to a branch belonging to the second part, and the node voltage E_a as the variables. In this way, we can obtain a set of the independent variables (i^p_0, E_a) of a network, which we shall call the *diakoptical coordinates*.

Diakoptics and codiakoptics will be shown to be methods for analyzing a network by the diakoptical coordinates. A remarkable feature is that, if partition of a network is done in such a way that all the branches are included in the first part, the diakoptical coordinates reduce to the mesh coordinates, and if all the branches are included in the second part, we obtain the node coordinates. Therefore, the diakoptical and the codiakoptical methods contain the ordinary mesh and node-pair methods as two extreme cases.

The number of variables necessary for the calculation of a network has been believed to be larger in the diakoptical analysis than in either the mesh or the node-pair analysis. But in our generalized diakoptical analysis in terms of the above-defined diakoptical coordinates, the number can be made even less than in either of the two ordinary methods. This will be shown from the topological structure of the dissection of a network in a later section (§7).

1·2. Fundamental equation of diakoptics and codiakoptics. Here we analyze a network by the diakoptical coordinates. First, we consider the second part dissected above in which the variables are E_a. The node-pair method will be used to analyze the part, considering the influence of currents i^p_0 flowing in the other part.[2] Viewed from the second part, the in-

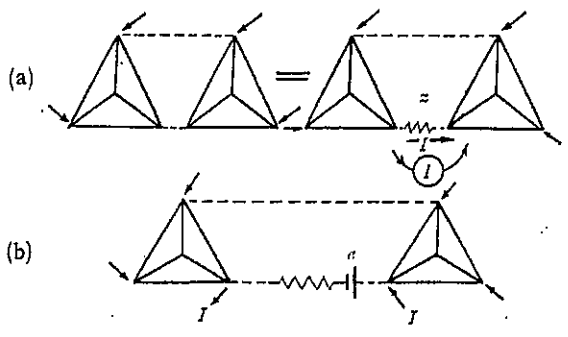


FIG. 2

1) The orthogonal coordinates (i^p, E_a), which consists of all the mesh currents and the node voltages include some redundant variables from the point of view of practical analysis.

2) The problem of the second part only does not generally constitute a self consistent problem, i. e. the sum of the impressed node-currents for each connected subnetwork of this part does not vanish. In such a case, let the superfluous currents flow through branches of the first part to another subnetwork, so that the sums of the impressed currents over each connected subnetwork vanish. After that, we add some voltage sources to the branches so as to eliminate the voltage drop due to the currents. This operation corresponds to the conversion of a part of the current sources to the equivalent voltage sources. See Fig. 2 where the sum of the impressed node currents is I in the right subnetwork and $-I$ in the left subnetwork of Fig. 2(a). We have equivalent impressed quantities shown Fig. 2(b), where the sum of the impressed node currents over each subnetwork vanishes and $e=ZI$.

fluence of i^{p}_{0} appears as a part of the superposed impressed node-current sources I^{a}_{0} (see Fig. 3).

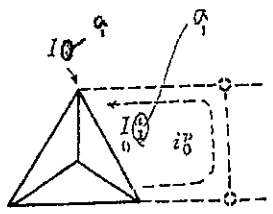


FIG. 3

We define here an *incidence matrix* $(K^{a}_{\frac{1}{p}})$ between the nodes of the second part and the meshes of the first part by

$$K^{a}_{\frac{1}{p}}=\begin{cases}-1, & \text{if the } \underset{0}{p}\text{-th mesh current flows into at the } \underset{1}{a}\text{-th node,}\\ +1, & \text{if the } \underset{0}{p}\text{-th mesh current flows out from the } \underset{1}{a}\text{-th node,}\\ 0, & \text{otherwise.}\end{cases} \quad (1\cdot9)^{1)}$$

Then the currents I^{a}_{0} flowing into the $\underset{1}{a}$-th node from the first part of the network are

$$I^{a}_{0}=-K^{a}_{\frac{1}{p}}i^{p}_{0}. \quad (1\cdot10)$$

The equation of the second part is written as

$$y^{ba}_{11}E_{a}=I^{b}_{1}-K^{b}_{\frac{1}{p}}i^{p}_{0}, \quad (1\cdot11)$$

where I^{b}_{1} are the impressed node-currents and (y^{ba}_{11}) is the node-admittance matrix of the second part only.[2)]

On the other hand, the influence of the second part appears as impressed voltage sources e_{q} seen from the viewpoint of the first part. These apparent voltage sources e_{q} are written

$$e_{q}=K^{a}_{\frac{1}{q}}E_{a}. \quad (1\cdot12)$$

Accordingly by the mesh method, the equation of the first part, taking account of the influence of the second part becomes

$$z_{qp}i^{p}=e_{q}+K^{a}_{\frac{1}{q}}E_{a} \quad (1\cdot13)$$

where e_{q} is the impressed e. m. f. of the e_{q}-th mesh and (z_{qp}) is the mesh-impedance matrix of the first part only.[3)]

The equation for the diakoptical coordinates $(i^{p}_{0},\ E_{a})$ is obtained by combining (1·11) and (1·13),

$$\begin{cases} y^{ab}_{11}E_{b}+K^{a}_{\frac{1}{p}}i^{p}_{0}=I^{a}_{1},\\ -K^{b}_{\frac{1}{q}}E_{b}+z_{qp}i^{p}=e_{q}, \end{cases} \quad (1\cdot14)$$

or in matric form

$$\begin{pmatrix} y^{ab}_{11} & K^{a}_{\frac{1}{p}}\\ -K^{b}_{\frac{1}{q}} & z_{qp}\end{pmatrix}\begin{pmatrix} E_{b}\\ i^{p}\end{pmatrix}=\begin{pmatrix} I^{a}\\ e_{q}\end{pmatrix}. \quad (1\cdot15)$$

If some impressed quantities are given as voltage sources in the second part and as current sources in the first part, each can easily be converted into the other kind of sources equivalently (see e. g. [23], [24], also footnote on p. 325).

We propose the system (1·14) or equation (1·15) as the "fundamental equation of diakoptics and codiakoptics", since all the procedures of diakoptics and codiakoptics can be derived from this, as will be seen later.

1·3. Diakoptical and codiakoptical procedures. The diakoptical procedures and the codiakoptical procedures are proved to be equivalent to solving the fundamental equation (1·15) by partitioning the coefficient matrix.

In the diakoptical procedures, we first invert the upper-left half y^{ab}_{11} of (1·15). This corresponds just to solving the problems of subnetworks. Then, we must invert the matrix

$$z_{qp}-K^{a}_{\frac{1}{p}}(y^{ab}_{11})^{-1}K^{b}_{\frac{1}{q}}.$$

1) $K^{a}_{\frac{1}{p}}$ can be defined by

$$K^{a}_{\frac{1}{p}}=D^{a}_{\frac{1}{k}}R^{k}_{\frac{0}{p}}=-D^{a}_{\frac{i}{1}}R^{i}_{\frac{k}{p}}.$$

2) If the second part is composed of several disjoint parts (two in the case of Fig. 1), (1·11) splits into as many independent groups of equations.

3) If the first part is composed of several disjoint parts (there is only one such part in the case of Fig. 1), (1·13) splits into as many independent groups of equations.

This procedure corresponds to solving the intersection network. In the diakoptical procedures, therefore, it is convenient to tear the network in such a manner that the second part of the network consists of several disconnected parts.

On the contrary, it is convenient for the codiakoptical procedures to tear the network in such a manner that the first part consists of several disconnected parts, for we have to start with the inversion of the lower part $z_{\substack{qp\\000}}$ of (1·15), and then we invert

$$y^{ab}_{11} - K^{a}_{1p}(z_{\substack{qp\\000}})^{-1}K^{b}_{1q}$$

to obtain the solution for the interconnected system.

In the diakoptical procedures, if the tearing is made in such a way that the first part does not contain any nodes (hence contains branches only called cut-branches), and each connected components of the second part have a common grounded point, the above procedure coincides exactly with Kron's approach [2]. On the other hand, if the second part does not contain any mesh, the codiakoptical procedures completely coincide with Onodera's approach [3].

However, being applicable to more general manners of tearing, our method may be said to be an extended one. It is sometimes effective to tear the network, for example in the diakoptical case, in such a manner that the intersection network, i. e. the first part of the network, contains some nodes.

In the following section, we shall clarify the intrinsic structures of the method using topological methods more thoroughly.

2. Dissection of Networks

Diakoptics and codiakoptics can be said to be a method of solving networks by means of network dissection. We shall study here the dissection of network complexes. Each subcomplex thus dissected also constitutes a complex with the relative topology of the entire complex. We shall deal with the mutual relations between these complexes, so that the mutual relations between the solution of the entire network and those of the subnetworks will be clarified.

2·1. Network complex. A network is composed of branches (1-cells from the topological viewpoint) denoted by $\sigma^1_\kappa(\kappa=1, 2, \ldots, n : n$ is the number of branches) and nodes (0-cells) denoted by $\sigma^0_a(a=1, 2, \ldots, m+1 : m+1$ is the number of nodes). It constitutes a 1-dimensional complex with the incidence relation expressed by the incidence matrix (cut-set matrix) D^a_κ. But it is more convenient from the dualistic viewpoint to deal with it as a 2-dimensional complex X by adding "meshes" (2-cells) in such a manner that every loop becomes the boundary of a 2-cells [24]. Then the incidence relation between meshes and branches is represented by the loop matrix $R^\lambda_p(\lambda=1, \ldots, n;\ p=1, \ldots, k : k$ is the number of independent meshes).

The superimposed physical quantities such as currents and voltages are represented by chains (or cochains) of a network complex X over a suitable coefficient domain. For instance, a current configuration of a network is represented by a 1-chain

$$C^1 = s^\kappa \sigma^1_\kappa,$$

and a voltage configuration by a 1-cochain

$$C_1 = u_\lambda \sigma^\lambda_1$$

where $s^\kappa(u_\kappa)$ denotes the current in (or voltage across) branch κ.

The network analysis can be done using these chains and their (co-)boundaries (see [24]), it can be said, in quite a topological manner.

2·2. Closed and open subcomplexes. We dissect a 2-dimensional network complex X into two parts $\overset{0}{X}$ and $\overset{1}{X}$. In our dissection of §1, the first part $\overset{0}{X}$ should satisfy the condition that all the meshes having at least a branch of $\overset{0}{X}$ as constituent elements should also be contained in $\overset{0}{X}$. More precisely, all the elements, of whose faces at least one belongs to $\overset{0}{X}$, should also belong to $\overset{0}{X}$. This means that $\overset{0}{St}\overset{0}{X}=\overset{0}{X}$, $(\overset{0}{St}\overset{0}{X}=\{\sigma' \mid \sigma' > \overset{0}{\sigma},\ \overset{0}{\sigma}\epsilon\overset{0}{X}\}$, and $\sigma' > \overset{0}{\sigma}$ means that $\overset{0}{\sigma}$ is a face of $\overset{0}{\sigma'}$). Such a part $\overset{0}{X}$ is said to be open, and $\overset{0}{X}$ constitutes a subcomplex of X with the relative topology [4]. Being the complement of $\overset{0}{X}$, the other part $\overset{1}{X}$ constitutes a closed subcomplex of X, i. e. all the elements which are faces of at least one element of $\overset{1}{X}$

also belong to X_1, or $ClX=X_1$, $(ClX_1=\{\sigma' | \sigma'<\sigma, \sigma\epsilon X_1)$.

The pair (X_1, X_0) is a dissection of X. 'The first part X_0 should be composed in such a way that, if X_0 contains a node it also contains all the branches incident to the node, and if X_0 contains a branch it also contains all the meshes incident to it. Its complement X_1 becomes closed, i. e. if X_1 contains a branch (mesh), it also contains all the nodes (branches) incident to it.

It must be noted that each X_i $(i=0,1)$ does not necessarily compose a connected subcomplex, since in diakoptics we have more than one closed subnetwork, of which X_1 consists. In Kron's diakoptical analysis, his cut-branches, with their incident meshes, constitute X_0. An example of the dissection is shown in Fig. 4, where the full lines represent the elements (nodes, branches and meshes) of X_1 and the dotted lines show the elements of X_0.

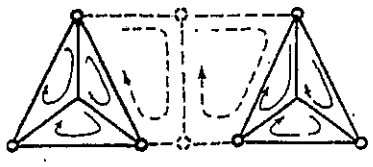


FIG. 4

The diakoptical coordinates may be said to consist of the (homologically) independent mesh currents (2-chain) in X_0 and of (cohomologically) independent node voltages (0-cochain) in X_1.

2·3. Projections to and injections from subcomplexes. Our aim is to solve a network problem on the original complex X with the help of subcomplexes X_1 and X_0, for it enables us to use the topological informations more completely than by any other method. Hence our present purpose is to find the relations between the groups of chains, cycles and boundaries of X_i over a suitable coefficient domain and those of X itself. To investigate these relations, four fundamental chain transformations π_i and ι_i $(i=0, 1)$ will be defined, and they will play important rôles, as follows.

The projection $\pi_i: C\to C_i$ is naturally defined regarding the X_i-part of a chain $C(\epsilon C)$ as a chain $C_i(\epsilon C_i)$ of X_i, where C is the group of chains of X and C_i is the group of chains of X_i $(i=0,1)$. We shall write

$$\pi_1 C=C_1, \quad \pi_0 C=C_0$$

or shortly

$$\pi_i C=C_i. \qquad (2\cdot1)$$

This operation π_i is called the "*projection of X onto X_i*". Physically speaking, to operate π_i on a chain C means to regard physical quantities (such as currents) represented by C as those of X_i.

Since the elements of X_i are among those of X, we have an injection $\iota_i: C_i\to C$ by regarding a chain of X_i as a chain of X itself. We call the operation ι_i the "*injection of X_i into X*". This means physically that we regard the physical quantities of X_i as those of X.

The following two relations are easily proved from the facts that $X=X_1+X_0$ and each X_i is the complement of the other.

$$\iota_1\pi_1+\iota_0\pi_0=1 \text{ (identity operator)}, \qquad (2\cdot2)$$

$$\pi_i\iota_j=\delta_{ij} \text{ (Kronecker's delta)}. \qquad (2\cdot3)$$

From (2), we have the following theorem:

Theorem 1. *No informations are lost by operating $\iota_1\pi_1+\iota_0\pi_0$ on a chain of X, that is, by projecting a chain of X into X_1 and X_0 respectively and then gathering them by the injections from both X_1 and X_0.*

This theorem shows the validity of using the dissection processes or "diakoptics" as a method to solve network problems. (2·3) means that X_1 and X_0 have no common element.

2·4. Boundary operator of X_i. In subcomplexes X_1 and X_0, the incidence relations

are the same as in X.[1] Therefore the boundary operators ∂ in X and ∂ in X are so defined that ∂C is the X-part of ∂C, where C is the injection of C into X. Thus we have

$$\underset{1}{\partial} \overset{\text{def}}{=} \pi \underset{1\ 1}{\partial} \iota, \quad \underset{0}{\partial} \overset{\text{def}}{=} \pi \underset{0\ 0}{\partial} \iota. \tag{2·4}$$

These can be rewritten in matric form using R_{β}^{κ} and D_{κ}^{α}

$$\underset{1}{\partial} = \begin{cases} R_{\beta}^{\kappa}: \text{ when operated on a 2-chain } \{s_1^{\beta}\}, \\ \quad (C^2 = s_1^{\beta}\sigma_{\beta}^2) \\ D_{\kappa}^{\alpha}: \text{ when operated on a 1-chain } \{s_1^{\kappa}\}, \end{cases}$$

$$\underset{0}{\partial} = \begin{cases} R_{\beta}^{\kappa}: \text{ when operated on a 2-chain } \{s_0^{\beta}\}, \\ D_{\kappa}^{\alpha}: \text{ when operated on a 1-chain } \{s_0^{\kappa}\}, \end{cases}$$

where $\underset{i}{\alpha}$, $\underset{i}{\kappa}$ and $\underset{i}{\beta}(i=0,1)$ denote the nodes, branches and meshes belonging to $\underset{i}{X}$. Since X is closed, $\partial(\underset{1\ 1}{\iota C})$ is always on $\underset{1}{X}$ only, so that we have

$$\underset{0\ 1}{\pi\partial}\iota = 0, \tag{2·5}$$

or

$$\begin{cases} R_{\beta}^{\kappa} = 0, \\ D_{\kappa}^{\alpha} = 0. \end{cases}$$

But X is not closed and $\partial(\underset{0\ 0}{\iota C})$ may have an $\underset{1}{X}$-part. Therefore we have a non-zero operator

$$\underset{1\ 0\ \ 10}{\pi\partial\iota} \overset{\text{def}}{=} \underset{10}{\partial}, \tag{2·6}$$

$$\underset{10}{\partial} = \begin{cases} R_{\beta}^{\kappa}: \text{ when operated on a 2-chain of } \underset{0}{X}, \\ D_{\kappa}^{\alpha}: \text{ when operated on a 1-chain of } \underset{0}{X}. \end{cases}$$

$\underset{10}{\partial}$ is an operator which transforms $\underset{0}{C}^r$ into $\underset{1}{C}^{r-1}$, consisting in taking the $\underset{1}{X}$-part of the boundary $\partial(\underset{0\ 0}{\iota C})$ of $\underset{0}{C}$ in X. Hence, $\underset{10}{\partial}$ represents the connexion relations of $\underset{1}{X}$ and $\underset{0}{X}$, and the influences of $\underset{0}{C}$ are carried over to $\underset{1}{C}$ through this operator, as we shall see later.

Dually to the above, the coboundary operators $\underset{0}{\delta}$ in $\underset{0}{X}$ and $\underset{1}{\delta}$ in $\underset{1}{X}$ are defined as follows,

$$\underset{0}{\delta} \overset{\text{def}}{=} \underset{000}{\pi\delta}\iota, \quad \underset{1}{\delta} \overset{\text{def}}{=} \underset{111}{\pi\delta}\iota, \tag{2·7}$$

or

$$\underset{i}{\delta} = \begin{cases} D_{\kappa}^{\alpha}: \text{ when operated on} \\ \quad \text{a 0-cochain of } \underset{i}{X}, \\ R_{\beta}^{\kappa}: \text{ when operated on} \\ \quad \text{a 1-cochain of } \underset{i}{X}. \end{cases}$$

Since $\underset{0}{X}$ is open, $\delta(\underset{00}{\iota C})$ has the $\underset{0}{X}$-part only, so that we have

$$\underset{1\ 0}{\pi\delta}\iota = 0 \tag{2·8}$$

or

$$\begin{cases} D_{\kappa}^{\alpha} = 0, \\ R_{\beta}^{\kappa} = 0. \end{cases}$$

But X is not open, and $\delta(\underset{11}{\iota C})$ has sometimes an $\underset{0}{X}$-part. Therefore, we have a non-zero

$$\underset{0\ 1\ \ 01}{\pi\delta\iota} = \underset{01}{\delta} \tag{2·9}$$

or

$$\underset{01}{\delta} = \begin{cases} D_{\kappa}^{\alpha}: \text{ when operated on} \\ \quad \text{a 0-cochain of } \underset{1}{X}, \\ R_{\beta}^{\kappa}: \text{ when operated on} \\ \quad \text{a 1-cochain of } \underset{1}{X}. \end{cases}$$

$\underset{01}{\delta}$ is an operator which transforms $\underset{1}{C}_r$ into $\underset{0}{C}_{r+1}$ and the influences of $\underset{1}{C}$ are carried over to $\underset{0}{C}$ through this operator.

Examples of these operators are as follows (see Fig. 5):

$$\underset{1}{\partial}\sigma_1^1 = \sigma_1^0 - \sigma_2^0, \qquad \underset{0}{\partial}\sigma_4^1 = \sigma_2^2,$$
$$\underset{0}{\partial}\sigma_2^2 = \sigma_4^1 + \sigma_5^1 + \sigma_6^1, \qquad \underset{1}{\partial}\sigma_3^1 = \sigma_1^2,$$

1) The boundary operator ∂ in X is represented in matrix form as follows;

$$\partial = \begin{cases} R_{\beta}^{\kappa}: \text{ when operated on a 2-chain } (i^{\beta}), & C^2 = i^{\beta}\sigma_{\beta}^2, \\ D_{\kappa}^{\alpha}: \text{ when operated on a 1-chain } (i^{\kappa}), & C^1 = i^{\kappa}\sigma_{\kappa}^1. \end{cases}$$

The coboundary operator δ is also represented by

$$\delta = \begin{cases} D_{\kappa}^{\alpha}: \text{ when operated on a 0-cochain } (E_{\alpha}), \\ \qquad C_0 = E_{\alpha}\sigma_0^{\alpha}, \\ R_{\beta}^{\kappa}: \text{ when operated on a 1-cochain } (E_{\kappa}), \\ \qquad C_1 = E_{\kappa}\sigma_1^{\kappa}. \end{cases}$$

$$\underset{10}{\partial}\,\sigma_4^1=\sigma_1^0, \qquad \underset{01}{\delta}\,\sigma_1^0=\sigma_4^1,$$
$$\underset{10}{\partial}\,\sigma_2^2=\sigma_3^1, \qquad \underset{01}{\delta}\,\sigma_3^1=\sigma_2^2.$$

In order to be sure whether these $\underset{i}{X}$'s with boundary operators $\underset{i}{\partial}$ ($\underset{i}{\delta}$) fulfil the conditions for a set to be a complex or not, the relation $\underset{ii}{\partial\partial}=0$ ($\underset{ii}{\delta\delta}=0$) needs to be proved.

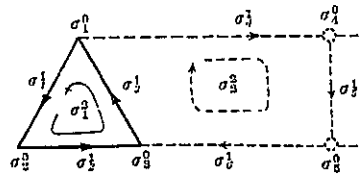


FIG. 5

Proof: From $\underset{11}{\iota\pi}+\underset{00}{\iota\pi}=1$ and $\underset{ij}{\pi\iota}=\delta_{ij}$, we have

$$\partial=\underset{11}{(\iota\pi}+\underset{00}{\iota\pi)}\,\partial\,(\underset{11}{\iota\pi}+\underset{00}{\iota\pi})$$
$$=\underset{111}{\iota\partial\pi}+\underset{1100}{\iota\,\partial\,\pi}+\underset{000}{\iota\partial\pi},$$
$$0=\partial\partial$$
$$=\underset{1111}{\iota\partial\partial\pi}+\underset{11100}{\iota\partial\,\partial\,\pi}+\underset{11000}{\iota\,\partial\,\partial\pi}$$
$$+\underset{0000}{\iota\partial\partial\pi}.$$

$$\delta=\underset{11}{(\iota\pi}+\underset{00}{\iota\pi)}\,\delta\,(\underset{11}{\iota\pi}+\underset{00}{\iota\pi})$$
$$=\underset{111}{\iota\delta\pi}+\underset{0011}{\iota\,\delta\,\pi}+\underset{000}{\iota\delta\pi},$$
$$0=\delta\delta$$
$$=\underset{0000}{\iota\delta\delta\pi}+\underset{00011}{\iota\,\delta\,\delta\,\pi}+\underset{00111}{\iota\,\delta\,\partial\pi}$$
$$+\underset{1111}{\iota\delta\delta\pi}.$$

Operating $\underset{i}{\iota}$ from the right and $\underset{i}{\pi}$ from the left, we have

$$\underset{00}{\partial\partial}=0, \quad \underset{11}{\partial\partial}=0. \qquad (2\cdot10)$$

$$\underset{11}{\delta\delta}=0, \quad \underset{00}{\delta\delta}=0. \qquad (2\cdot11)$$

We see that the operators $\underset{1}{\iota}$ and $\underset{1}{\pi}$ commute with the boundary and coboundary operators:

$$\left.\begin{aligned}&\underset{1}{\partial}\,\underset{11}{\iota}-\underset{11}{\iota}\,\underset{00}{\partial}=\underset{11}{(\iota\pi}+\underset{00}{\iota\pi)}\,\partial\,\underset{1}{\iota}-\underset{11}{\iota\pi}\,\partial\,\underset{1}{\iota}=\underset{11}{\iota\pi}\,\partial\,\underset{1}{\iota}=0,\\ &\underset{1}{\pi}\,\partial-\partial\,\underset{11}{\pi}=\underset{1}{\pi}\,\partial\,(\underset{11}{\iota\pi}+\underset{00}{\iota\pi})-\underset{1}{\pi}\,\partial\,\underset{11}{\iota\pi}=\underset{1}{\pi}\,\partial\,\underset{11}{\iota\pi}=0,\end{aligned}\right\} \qquad (2\cdot12)$$

and that $\underset{1}{\iota}$ and $\underset{1}{\pi}$ are dual mappings,

$$\underset{1}{\iota}^*=\underset{1}{\pi}, \quad \underset{1}{\pi}^*=\underset{1}{\iota}.$$

Therefore, $(\underset{1}{\iota}, \underset{1}{\pi})$ is a pair of dual chain-mappings.[1] In the same way, we have

$$\underset{00}{\partial\pi}-\underset{0}{\pi}\,\partial=0, \quad \underset{0}{\delta\iota}-\iota\,\underset{00}{\delta}=0$$

and $\underset{0}{\pi}^*=\underset{0}{\iota}$, $\underset{0}{\iota}^*=\underset{0}{\pi}$. Hence, $(\underset{0}{\pi}, \underset{0}{\iota})$ is another pair of dual chain-mappings.

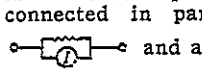
3. Network Analysis by Dissection (1)—*No mutual couplings existing between* $\underset{1}{X}$ *and* $\underset{0}{X}$.

3·1. Problem. The problem of linear electric networks can be stated as follows:

> "Given current sources $D^1=I^\kappa\sigma_\kappa^1$[2] and voltage sources $D_1=e_\lambda\sigma_1^\lambda$, we require $C^1=i^\kappa\sigma_\kappa^1$ and $C_1=E_\lambda\sigma_1^\lambda$, such that $\partial(C^1-D^1)=0$ ($D_\kappa^\alpha(i^\kappa-I^\kappa)=0$) and $\partial(C_1-D_1)=0$ ($R_p^\lambda(E_\lambda-e_\lambda)=0$): Kirchhoff's laws, and $i^\kappa=y^{\kappa\lambda}E_\lambda$ or $E_\lambda=Z_{\lambda\kappa}i^\kappa$: Ohm's law.

1) A chain transformation, $\tau: X\to Y$, $\tau x_i^p=a_i^j(p)y_j^p$ is called a chain mapping, if τ commutes with the boundary operator, $\tau\partial-\partial\tau=0$. The dual mapping $\tau^*: Y^*\to X^*$, of τ is defined by $\tau^*y_p^j=a_i^j(p)x_p^i$.

2) If node current sources $D^0=I^\alpha\sigma_\alpha^0$ and mesh voltage sources $D_2=e_\varrho\sigma_2^\varrho$ are given, we can reduce these to equivalent branch current sources $D^1=I^\kappa\sigma_\kappa^1$ and voltage sources $D_1=e_\lambda\sigma_1^\lambda$ where $I^\alpha=D_\kappa^\alpha I^\kappa$, $e_p=R_p^\kappa e_\kappa$ holds.

For example, we can put $I^\kappa=R_\alpha^\kappa I^\alpha$, $e_\lambda=D_\lambda^\varrho e_\varrho$, where (R_α^κ) and (D_λ^ϱ) are a tree matrix and a cotree matrix [24], respectively. A current source is to be connected in parallel with a branch such as  and a voltage source is to be connected in series with a branch such as .

We know already two methods of solving the above problem, one concerning the mesh characteristics of the network and the other concerning the node characteristics. The third method by dissection, now to be proposed, uses both methods simultaneously. Before describing the method, we assume the following two simplifications.

1. *The given network can so be dissected that, in the first place, the given current sources* $D^1=I^\kappa\sigma^1_\kappa$ *lie on the* $\underset{1}{X}$*-part only, and the given voltage sources* $D_1=e_\lambda\sigma^\lambda_1$ *lie on the* $\underset{0}{X}$*-part only, i. e.*

$$\underset{0}{\pi}D^1=0,\quad \underset{1}{\pi}D_1=0. \tag{3·1}$$

As we can convert branch current sources into the equivalent branch voltage sources, and vice versa[1], this simplification can always be satisfied, so that it does not impose any essential restrictions on our method.

2. *In the second place, there are no mutual admittances (impedances) between any branches of* $\underset{1}{X}$ *and any branches of* $\underset{0}{X}$.

Since $y^{\kappa\lambda}$ can be regarded as a mapping $y: C_1=\{E_\lambda\}\to C^1=\{i^\kappa\}$, we may write this assumption as

$$\left.\begin{aligned}&\underset{1}{\pi}y_\iota\underset{0}{\pi}=0,\quad \underset{0}{\pi}y_\iota\underset{1}{\pi}=0,\\ &(\underset{1}{\pi}z_\iota\underset{0}{\pi}=0,\quad \underset{0}{\pi}z_\iota\underset{1}{\pi}=0).\end{aligned}\right\} \tag{3·2}$$

We define the admittance and impedance matrices of $\underset{i}{X}$ as

$$\underset{i}{y}=\underset{i}{\pi}y_\iota\underset{i}{\pi},\quad \underset{i}{z}=\underset{i}{\pi}z_\iota\underset{i}{\pi}, \tag{3·3}$$

and we see $\underset{i}{y}\cdot\underset{i}{z}=\underset{i}{z}\cdot\underset{i}{y}=\mathbf{1}$ (unit matrix), for

$$\underset{i}{y}\cdot\underset{i}{z}=\underset{i}{\pi}y_\iota\underset{\iota\iota}{\pi}z_\iota\underset{\iota}{\pi}=\underset{i}{\pi}yz_\iota\underset{\iota}{\pi}-\underset{i}{\pi}y_\iota\underset{\iota\iota}{\pi}z_\iota\underset{\iota}{\pi}\quad(i\neq j)$$
$$=\underset{ii}{\pi}{}_\iota=1.$$

we obtain

$$\left.\begin{aligned}y&=(\underset{11}{\iota\pi}+\underset{00}{\iota\pi})y(\underset{11}{\iota\pi}+\underset{00}{\iota\pi})\\ &=\underset{111}{\iota y\pi}+\underset{000}{\iota y\pi},\\ z&=\underset{111}{\iota z\pi}+\underset{000}{\iota z\pi}.\end{aligned}\right\} \tag{3·4}$$

In matric forms, these are represented as

$$y=y^{\kappa\lambda}=\begin{array}{c}\\ \overset{X}{\underset{1}{}}\\ \overset{X}{\underset{0}{}}\end{array}\begin{array}{c}\overset{\overset{X}{\underset{1}{}}}{\overset{\overset{X}{\underset{0}{}}}{}}\\ \left(\begin{array}{c|c}\underset{1}{y^{\kappa\lambda}_{11}} & 0\\ \hline 0 & \underset{0}{y^{\kappa\lambda}_{00}}\end{array}\right)\end{array},$$

$$z=z_{\lambda\kappa}=(y^{\kappa\lambda})^{-1}=\begin{array}{c}\\ \overset{X}{\underset{1}{}}\\ \overset{X}{\underset{0}{}}\end{array}\begin{array}{c}\overset{\overset{X}{\underset{1}{}}}{\overset{\overset{X}{\underset{0}{}}}{}}\\ \left(\begin{array}{c|c}\underset{111}{z_{\lambda\kappa}} & 0\\ \hline 0 & \underset{000}{z_{\lambda\kappa}}\end{array}\right)\end{array}$$

$$=\begin{array}{c}\\ \overset{X}{\underset{1}{}}\\ \overset{X}{\underset{0}{}}\end{array}\begin{array}{c}\overset{\overset{X}{\underset{1}{}}}{\overset{\overset{X}{\underset{0}{}}}{}}\\ \left(\begin{array}{c|c}\underset{1}{(y)^{-1}} & 0\\ \hline 0 & \underset{0}{(y)^{-1}}\end{array}\right)\end{array}.$$

In ordinary cases, mutual couplings, even if they exist, concentrate only locally, so that this assumption will be satisfied in most cases, if we dissect (tear) the network appropriately. The more general case in which this assumption is not satisfied will be dealt with in the next section, by extending the method.

3·2. Fundamental equation of diakoptics and codiakoptics. Under the above assumptions, we introduce the fundamental equation of our method as follows:

Taking account of

Kirchhoff's 2nd law

$$\partial(C_1-D_1)=0$$
$$(R^\kappa_\beta(E_\kappa-e_\kappa)=0),$$

Kirchhoff's 1st law

$$\partial(C^1-D^1)=0$$
$$(D^\alpha_\lambda(i^\lambda-I^\lambda)=0),$$

1) For given current sources $D^1=I^\kappa\sigma^1_\kappa$ we have $D'_1=e'_\lambda\sigma^\lambda_1=(z_{\lambda\kappa}I^\kappa)\sigma^\lambda_1$, as the equivalent voltage sources, and conversely, for given voltage sources $D_1=e_\lambda\sigma^\lambda_1$, we have $D^{1\prime}=I'^\kappa\sigma^1_\kappa=(y^{\lambda\kappa}e_\lambda)\sigma^1_\kappa$ as the equivalent current sources.

we can put

$$C_1 - D_1 = \delta C_0$$
$$(E_\kappa - e_\kappa = D^a_\kappa E_a).$$

Operating on this by $\underset{1}{\pi}$ we have

$$\underset{1}{C}_1 = \underset{1}{\pi}\delta C_0 = \underset{11}{\delta} C_0 \qquad (+3\cdot5)$$
$$(\underset{11}{E}_\kappa = D^a_\kappa \underset{1}{E}_a),$$

for $\underset{1}{\pi} D_1 = 0$ and $\underset{1}{\pi}\delta = \underset{11}{\delta}\pi$. This is Kirchhoff's 2nd law in $\underset{1}{X}$.

Next, from $\underset{1}{\pi}\delta(C^1 - D^1) = 0$, we have

$$\underset{1}{\delta}(\underset{1}{C}^1 - \underset{1}{D}^1) = -\underset{100}{\delta} C^1 \qquad (+3\cdot6)^{1)}$$
$$(D^a_\kappa (i^\kappa - I^\kappa) = -D^a_\lambda i^\lambda)$$

using $\underset{1}{\pi}\delta = \underset{11}{\delta}\pi + \underset{100}{\delta}\pi$ and $\underset{0}{\pi} D^1 = 0$. This is Kirchhoff's 1st law in $\underset{1}{X}$. Since $y = \underset{111}{\iota}\pi + \underset{000}{\iota}\pi$, Ohm's law in $\underset{1}{X}$ takes the form

$$\underset{1}{C}^1 = y \underset{11}{C}_1,$$

or

$$i^\kappa = y^{\kappa\lambda} \underset{1}{E}_\lambda. \qquad (+3\cdot7)$$

Substituting (+3·7) in (+3·6) and using (+3·5), we have

$$\underset{1111}{\partial y \partial} C_0 = \underset{11}{\partial} D^1 - \underset{100}{\partial} C^1 \qquad (+3\cdot8)$$
$$(D^a_\kappa y^{\kappa\lambda} D^b_\lambda \underset{1}{E}_b = D^a_\kappa I^\kappa - D^a_\kappa i^\kappa).$$

Combining (±3·8) and substituting (±3·5) in it, we obtain the fundamental equation of diakoptics and codiakoptics

$$\left.\begin{aligned} \underset{1111}{\partial y \partial} C_0 + \underset{1000}{\partial}\partial C^2 &= \underset{11}{\partial} D^1, \\ \underset{0111}{\partial}\partial C_0 + \underset{0000}{\partial z \partial} C^2 &= \underset{00}{\partial} D_1. \end{aligned}\right\} \qquad (3\cdot9)$$

The fundamental equation (3·9) is rewritten in the matrix form

1) $\underset{1}{\pi}\delta - \underset{11}{\partial}\pi = \underset{1}{\pi}\delta(\underset{11}{\iota}\pi + \underset{00}{\iota}\pi) - \underset{1}{\pi}\partial\underset{11}{\iota}\pi = \underset{1}{\pi}\delta\underset{11}{\iota}\pi = \underset{100}{\partial}\pi.$

we can put

$$C^1 - D^1 = \partial C^2$$
$$(i^\lambda - I^\lambda = R^\lambda_\rho i^\rho).$$

Operating on this by $\underset{0}{\pi}$ we have

$$\underset{0}{C}^1 = \underset{0}{\pi}\partial C^2 = \underset{00}{\partial} C^2 \qquad (-3\cdot5)$$
$$(\underset{0}{i}^\lambda = R^\lambda_\rho \underset{0}{i}^\rho),$$

for $\underset{0}{\pi} D^1 = 0$ and $\underset{0}{\pi}\partial = \underset{00}{\partial}\pi$. This is Kirchhoff's 1st law in $\underset{0}{X}$.

Next, from $\underset{0}{\pi}\delta(C_1 - D_1) = 0$, we have

$$\underset{0}{\delta}(\underset{0}{C}_1 - \underset{0}{D}_1) = -\underset{011}{\delta} C_1 \qquad (-3\cdot6)^{2)}$$
$$(R^\kappa_\rho (\underset{0}{E}_\kappa - \underset{0}{e}_\kappa) = -R^\kappa_\lambda \underset{1}{E}_\kappa)$$

using $\underset{0}{\pi}\delta = \underset{00}{\partial}\pi + \underset{011}{\delta}\pi$ and $\underset{1}{\pi} D_1 = 0$. This is Kirchhoff's 2nd law in $\underset{0}{X}$. Since $z = \underset{111}{\iota}z\pi + \underset{000}{\iota}z\pi$, Ohm's law in $\underset{0}{X}$ takes the form

$$\underset{0}{C}_1 = z \underset{00}{C}^1,$$

or

$$E_\lambda = Z_{\lambda\kappa} \underset{00}{i}^\kappa. \qquad (-3\cdot7)$$

Substituting (−3·7) in (−3·6) and using (−3·5), we have

$$\underset{0000}{\partial z \partial} C^2 = \underset{00}{\partial} D_1 - \underset{011}{\partial} C_1 \qquad (-3\cdot8)$$
$$(R^\lambda_\rho \underset{0}{z}_{\lambda\kappa} R^\kappa_\sigma \underset{0}{i}^\rho = R^\kappa_\sigma \underset{0}{e}_\kappa - R^\kappa_\lambda \underset{1}{E}_\kappa).$$

$$D^a_\lambda y^{\kappa\lambda} D^b_\lambda \underset{1}{E}_b + D^a_\kappa R^\kappa_\sigma \underset{0}{i}^\sigma = D^a_\kappa I^\kappa,$$
$$R^\lambda_\sigma D^b_\lambda \underset{1}{E}_b + R^\lambda_\rho \underset{0}{z}_{\lambda\kappa} R^\kappa_\sigma \underset{0}{i}^\sigma = R^\lambda_\rho \underset{0}{e}_\lambda.$$

From $\partial\partial = 0$ or $D^a_\kappa R^\kappa_\sigma = 0$, we obtain

$$0 = D^a_\kappa R^\kappa_\sigma = D^a_\kappa R^\kappa_\sigma + D^a_\kappa R^\kappa_\sigma. \qquad (3\cdot10)^{3)}$$

2) $\underset{0}{\pi}\delta - \underset{00}{\partial}\pi = \underset{0}{\pi}\delta(\underset{11}{\iota}\pi + \underset{00}{\iota}\pi) - \underset{0}{\pi}\delta\underset{00}{\iota}\pi = \underset{0}{\pi}\delta\underset{011}{\iota}\pi = \underset{011}{\delta}\pi.$

3) This relation is also obtained by operator calculus:

$$0 = \underset{1}{\pi}\underset{0}{\partial}\partial\underset{1}{\iota} = \underset{1}{\pi}\underset{0}{\partial}(\underset{11}{\iota}\pi + \underset{00}{\iota}\pi)\underset{0}{\partial}\underset{1}{\iota} = \underset{110}{\partial}\partial + \underset{100}{\partial}\partial,$$

and, in a similar way, we also have $0 = \underset{001}{\partial}\partial + \underset{011}{\partial}\partial$.

Defining $K_{\overset{1}{q}}^{\overset{a}{1}}=D_{\overset{1}{\kappa}}^{\overset{a}{1}}R_{\overset{0}{q}}^{\overset{\kappa}{0}}=-D_{\overset{1}{\kappa}}^{\overset{a}{1}}R_{\overset{0}{q}}^{\overset{\kappa}{1}}$, we interpret $K_{\overset{1}{q}}^{\overset{\kappa}{1}}$ as the incidence number between a node $\underset{1}{a}$ of $\underset{1}{X}$ and a mesh $\underset{0}{q}$ of $\underset{0}{X}$. Of course, the element of $K_{\overset{1}{q}}^{\overset{a}{1}}$ are simple integers 0, ±1. The mutual influence between $C_{\underset{1}{0}}(E_{\underset{1}{b}})$ and $C_{\underset{0}{2}}(i^{\overset{\beta}{0}})$ are represented by means of this matrix $K_{\overset{0}{q}}^{\overset{a}{1}}$ (or the contragradient operators $\partial\partial_{100}$ and $\delta\delta_{001}$). Putting

$$\left.\begin{aligned} &y_{\underset{1}{1}}^{\overset{ab}{11}}=D_{\overset{1}{\kappa}}^{\overset{a}{1}}y_{\underset{1}{1}}^{\overset{\kappa\lambda}{11}}D_{\underset{1}{\lambda}}^{\overset{b}{1}},\quad z_{\underset{000}{pq}}=R_{\overset{0}{p}}^{\overset{\lambda}{0}}z_{\underset{00}{\lambda\kappa}}R_{\overset{0}{q}}^{\overset{\kappa}{0}},\\ &I^{\overset{a}{1}}=D_{\overset{1}{\kappa}}^{\overset{a}{1}}I^{\overset{\kappa}{1}}\quad\text{and}\quad e_{\underset{0}{q}}=R_{\overset{0}{q}}^{\overset{\lambda}{0}}e_{\underset{0}{\lambda}}, \end{aligned}\right\}\quad(3\cdot11)$$

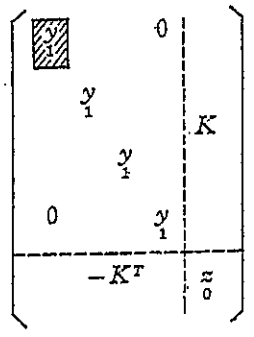
the fundamental equation (3·9) is rewritten as follows.

The fundamental equation of diakoptics and codiakoptics can be represented by the matric form

$$\left(\begin{array}{c|c} y_{\underset{1}{1}}^{\overset{ab}{11}} & K_{\overset{0}{p}}^{\overset{a}{1}} \\ \hline -K_{\overset{0}{q}}^{\overset{b}{1}} & z_{\underset{000}{qp}} \end{array}\right)\left(\begin{array}{c} E_{\underset{1}{b}} \\ \hline i^{\overset{\beta}{0}} \end{array}\right)=\left(\begin{array}{c} I^{\overset{a}{1}} \\ \hline e_{\underset{0}{q}} \end{array}\right).\quad(3\cdot12)$$

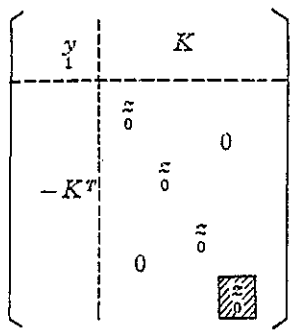
The $E_{\underset{1}{b}}$'s and $i^{\overset{\beta}{0}}$'s constitute the diakoptics coordinates of the network. We can solve (3·12) by partitioning the coefficient matrix. In this method, $K_{\overset{0}{p}}^{\overset{a}{1}}$ is so simple that we need little more than to solve the problems of each subcomplex $\underset{1}{X}$. This is the method of diakoptics and codiakoptics.

3·3. Diakoptics and Codiakoptics. Usually we dissect X in such a way that $\underset{1}{X}$ is composed of several disconnected subdivisions and that no mutual couplings exist between them, or that $\underset{0}{X}$ is composed of several disconnected subdivisions and no mutual couplings exist between them. In the former case the coefficient matrix of (3·12) assumes such a simple form as



where $\underset{1}{y}$ denotes a minor node admittance matrix corresponding to a disconnected subnetwork of $\underset{1}{X}$. Therefore, it is convenient to solve (3·12) by partitioning, beginning practically with the inversion of $y_{\underset{1}{1}}^{\overset{ba}{11}}$.[1] This procedure is the "diakoptics".

In the latter case, it is more convenient to begin with the inversion of $z_{\underset{000}{pq}}$ in solving (3·12) by partitioning, since the matrix becomes such a simple form as



where $\underset{0}{\approx}$ denotes a minor mesh-impedance matrix corresponding to a disconnected subnetwork of $\underset{0}{X}$. This procedure is the "codiakoptics" [3].

In these cases the amount of labour of inversion by partitioning the matrix is far smaller than ordinary method of the mesh type or node-pair type analysis. This is because we utilize the topological informations of the network in order to shorten the sequence of our procedures, by first dissecting it into $\underset{1}{X}$ and $\underset{0}{X}$.

In the above explanation, it has become clear that dokaiptics and codiakoptics are two dual methods and their fundamental equation basically the same, either (3·9) or (3·12). The only difference is that in partitioning we begin with the inversion of $y_{\underset{1}{1}}^{\overset{ba}{11}}$ in the former case, and with the inversion of $z_{\underset{000}{pq}}$ in the latter case.

More detailed procedures of calculation by

1) The inverse of a matrix

$$M=\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

by partitioning beginning with the inversion, of the minor matrix A, is written in the partitioned form as

$$M^{-1}=\left[\begin{array}{c|c} A^{-1}(I+BSCA^{-1}) & -A^{-1}BS \\ \hline -SCA^{-1} & S \end{array}\right]$$

where $S=(D-CA^{-1}B)^{-1}$ and I is the unit matrix.

dissection, i. e. diakoptics and codiakoptics, will be given in section 5. Before so doing we shall first investigate the possibility of extending the method to the more general case in which some branch-couplings exist between $\underset{1}{X}$ and $\underset{0}{X}$.

3·4. Examples.

Example 1: *Write down the fundamental equation of the network shown in Fig. 6.*

We can directly write down the fundamental equation of diakoptics (the number of variables is 8 in the diakoptical method whereas that by the node-pair method is 9):

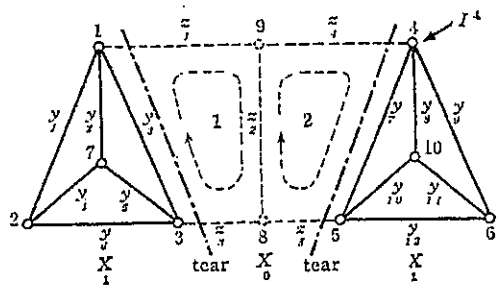


FIG. 6

$$\left(\begin{array}{ccc|ccc|cc|cc}
y_1+y_2+y_3 & -y_1 & -y_3 & & & & 1 & & E_1 & I^1 \\
-y_1 & y_1+y_4+y_6 & -y_6 & & 0 & & & & E_2 & I^2 \\
-y_3 & -y_6 & y_3+y_5+y_6 & & & & -1 & & E_3 & I^3 \\
\hline
 & & & y_7+y_8+y_9 & -y_7 & -y_9 & & -1 & E_4 & I^4 \\
 & 0 & & -y_7 & y_9+y_{10}+y_{11} & -y_{12} & & 1 & E_5 & I^5 \\
 & & & -y_9 & -y_{12} & y_9+y_{11}+y_{12} & & & E_6 & I^6 \\
\hline
-1 & & 1 & & & & z_1+z_2+z_3 & -z_2 & i^1 & e_1 \\
 & & & 1 & & -1 & -z_2 & z_2+z_4+z_6 & i^2 & e_2
\end{array}\right) = \left(\begin{array}{c} I^1 \\ I^2 \\ I^3 \\ I^4 \\ I^5 \\ I^6 \\ \hline e_1 \\ e_2 \end{array}\right),$$

where I^{α}_{i} is the applied node current and e_{β} is the applied mesh e. m. f., and (E_{α}, i^{β}) are the diakoptical coordinates of the network, the blanks in the matrix meaning 0.

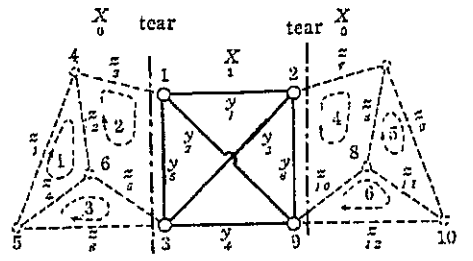


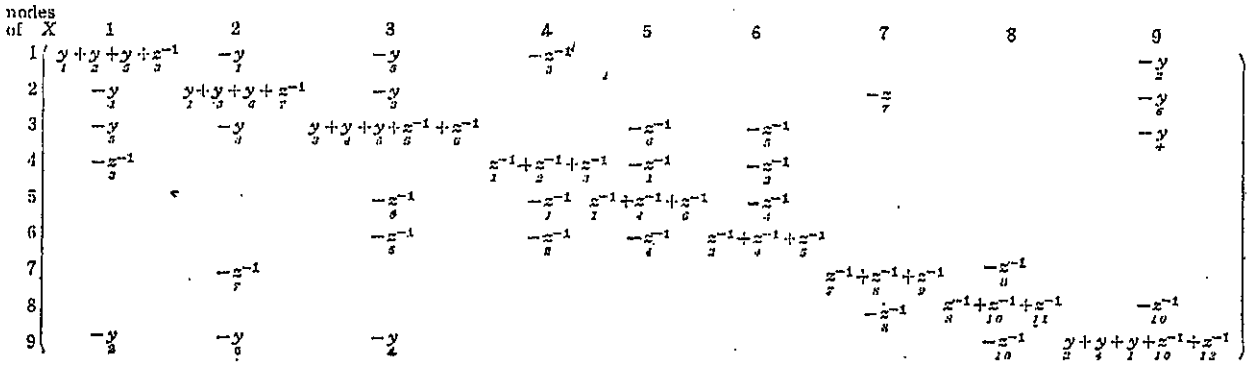
FIG. 7

Example 2 *Write down the fundamental equation of codiakoptics of the network shown in Fig. 7, and then compare it with the ordinary node admittance matrix.*

The coefficient matrix of the fundamental equation of codiakoptics is

		1	2	3	1	2	3	4	5	6
nodes of $\underset{1}{X}$	1	$y_1+y_2+y_5$	$-y_1$	$-y_5$		-1				
	2	$-y_1$	$y_1+y_3+y_6$	$-y_3$				1		
	3	$-y_5$	$-y_3$	$y_3+y_4+y_6$		1				
meshes of $\underset{0}{X}$	1				$z_1+z_2+z_4$	$-z_2$	$-z_4$			
	2	1		-1	$-z_2$	$z_2+z_3+z_5$	$-z_5$		0	
	3				$-z_4$	$-z_5$	$z_4+z_5+z_6$			
	4		-1					$z_7+z_8+z_{10}$	$-z_8$	$-z_{10}$
	5					0		$-z_8$	$z_8+z_9+z_{11}$	$-z_{11}$
	6							$-z_{10}$	$-z_{11}$	$z_{10}+z_{11}+z_{12}$

The node-admittance matrix has a more complicated form as



4. Network Analysis by Dissection (2)—*Mutual couplings existing between* $\overset{1}{X}$ *and* $\overset{0}{X}$

4·1. Problem and the mixed expression of Ohm's law. In the general case, the problem of networks can also be stated in the same manner as in subsection 3·1. Let us also assume that

$$\pi \overset{1}{D}{}^{1}=0 \text{ and } \pi \overset{1}{D}_{1}=0, \qquad (4\cdot1)$$

that is, the given current sources $\overset{1}{D}{}^{1}=I^{\kappa}\sigma^{1}_{\kappa}$ lie on the $\overset{0}{X}$-part only and the given voltage sources $\overset{1}{D}_{1}=e_{\lambda}\sigma^{\lambda}_{1}$ lie on the $\overset{0}{X}$-part only. Such a dissection is feasible without loss of generality and it does not impose any restriction on our method.

Since there are mutual admittances (impedances) existing not only between branches of $\overset{1}{X}$ and between those of $\overset{0}{X}$, but also between branches of $\overset{1}{X}$ and branches of $\overset{0}{X}$, we have non-zero operators,

$$\overset{\text{def}}{\underset{10}{y}}=\pi y_{\iota}\,\underset{1}{}\,\underset{0}{}, \qquad \overset{\text{def}}{\underset{01}{y}}=\pi y_{\iota}\,\underset{0}{}\,\underset{1}{},$$

$$\overset{\text{def}}{\underset{10}{z}}=\pi z_{\iota}\,\underset{1}{}\,\underset{0}{} \text{ and } \overset{\text{def}}{\underset{01}{z}}=\pi z_{\iota}\,\underset{0}{}\,\underset{1}{},$$

and we cannot proceed as in the previous section §3. Hence, we have to utilize here an alternative representation of Ohm's law.

Ohm's law represents the relations between the voltages across and the currents in the branches as the characters of the branches themselves, which have nothing to do with the connexion relations. As the admittance and impedance representations for Ohm's law, we already have

$$\left.\begin{aligned} &\overset{1}{C}{}^{1}=y\overset{1}{C}_{1} \\ &\text{or} \\ &\begin{pmatrix}\overset{1}{C}{}^{1}\\ \overset{0}{C}{}^{1}\end{pmatrix}=\begin{pmatrix}\underset{1}{y} & \underset{10}{y}\\ \underset{01}{y} & \underset{0}{y}\end{pmatrix}\begin{pmatrix}\overset{1}{C}_{1}\\ \overset{0}{C}_{1}\end{pmatrix},\quad \begin{pmatrix}\overset{\kappa}{i}{}^{1}\\ \overset{\kappa}{i}{}^{0}\end{pmatrix}=\begin{pmatrix}\overset{\kappa\lambda}{y}_{11} & \overset{\kappa\lambda}{y}_{10}\\ \overset{\kappa\lambda}{y}_{01} & \overset{\kappa\lambda}{y}_{00}\end{pmatrix}\begin{pmatrix}\overset{\lambda}{E}_{1}\\ \overset{\lambda}{E}_{0}\end{pmatrix} \end{aligned}\right\} \qquad (4\cdot2)$$

(admittance representation)

and

$$\overset{1}{C}_{1}=z\overset{1}{C}{}^{1}. \qquad (4\cdot3)$$

(impedance representation)

Here, we use the mixed representation of Ohm's law such as

$$\left.\begin{aligned} &\begin{pmatrix}\overset{1}{C}{}^{1}\\ \hdashline \overset{0}{C}_{1}\end{pmatrix}=\left(\begin{array}{c:c}\underset{1}{Y} & \underset{10}{\alpha}\\ \hdashline \underset{01}{\mu} & \underset{0}{Z}\end{array}\right)\begin{pmatrix}\overset{1}{C}_{1}\\ \hdashline \overset{0}{C}{}^{1}\end{pmatrix} \\ &\text{or} \\ &\begin{pmatrix}\overset{\kappa}{i}{}^{1}\\ \hdashline \overset{\lambda}{E}_{0}\end{pmatrix}=\left(\begin{array}{c:c}\overset{\kappa\lambda}{Y}_{11} & \overset{\kappa}{\alpha}{}^{1}_{100}\\ \hdashline \overset{\lambda}{\mu}{}^{1}_{010} & \overset{\lambda\kappa}{Z}_{000}\end{array}\right)\begin{pmatrix}\overset{\lambda}{E}_{1}\\ \hdashline \overset{\kappa}{i}{}^{0}\end{pmatrix} \end{aligned}\right\} \qquad (4\cdot4)$$

where the Y's have the dimensions of admittance, the $\overset{1}{Z}$'s the dimensions of impedance,

and the α's and μ's are non-dimensional.

Let us express these constants $\underset{1}{Y}$, $\underset{10}{\alpha}$, $\underset{01}{\mu}$ and $\underset{0}{Z}$ by means of the impedances z or the admittances y. From (4·2) and (4·4), we have

$$\begin{pmatrix} \underset{1}{Y} & \underset{10}{\alpha} \\ \underset{01}{\mu} & \underset{0}{Z} \end{pmatrix} = \begin{pmatrix} \underset{1}{y} - \underset{100}{y}\underset{0}{y^{-1}}\underset{01}{y} & \underset{10}{y}\underset{0}{y^{-1}} \\ -\underset{0}{y^{-1}}\underset{01}{y} & \underset{0}{y^{-1}} \end{pmatrix}. \quad (4\cdot5)$$

In a similar way, we obtain another expression

$$\begin{pmatrix} \underset{1}{Y} & \underset{10}{\alpha} \\ \underset{01}{\mu} & \underset{0}{Z} \end{pmatrix} = \begin{pmatrix} \underset{1}{z^{-1}} & -\underset{1}{z^{-1}}\underset{10}{z} \\ \underset{011}{z}\underset{1}{z^{-1}} & \underset{0}{z} - \underset{011}{z}\underset{1}{z^{-1}}\underset{10}{z} \end{pmatrix}. \quad (4\cdot6)$$

It must be noted that the mixed expression of Ohm's law or the branch characteristics is not merely theoretical but tangible. We can measure the constants $\underset{1}{Y}$, $\underset{10}{\alpha}$, $\underset{01}{\mu}$ and $\underset{0}{Z}$ directly, and this needs as much labour as measuring the constants $z(\underset{1}{z}, \underset{10}{z}, \underset{01}{z}$ and $\underset{0}{z})$ or y.

4·2. Fundamental equation of diakoptics and codiakoptics in the general case. All preparation now having been made, let us solve the equation. In the same way as in subsection 3·2, Kirchhoff's laws are rewritten as follows.

$$\underset{1}{C_1} = \partial \underset{11}{C_0}, \quad (+3\cdot5) \qquad \underset{0}{C^1} = \partial \underset{00}{C^2}, \quad (-3\cdot5)$$

$$\partial \underset{11}{C^1} + \partial \underset{100}{C^1} = \partial \underset{11}{D^1}. \quad (+3\cdot6) \qquad \partial \underset{00}{C_1} + \partial \underset{011}{C_1} = \partial \underset{00}{D_1}. \quad (-3\cdot6)$$

But in this case, Ohm's law is

$$\underset{1}{C^1} = \underset{11}{Y}\underset{1}{C_1} + \underset{100}{\alpha}\underset{0}{C^1}. \quad (+4\cdot7) \qquad \underset{0}{C_1} = \underset{011}{\mu}\underset{1}{C_1} + \underset{00}{Z}\underset{0}{C^1}. \quad (-4\cdot7)$$

Hence, substituting (±4·7) in (±3·6) and using (±3·5), we obtain

$$\partial \underset{1111}{Y}\partial \underset{1}{C_0} + (\partial + \partial \underset{110}{\alpha})\partial \underset{00}{C^2} = \partial \underset{11}{D^1}. \quad (+4\cdot8) \qquad \partial \underset{0000}{Z}\partial \underset{01}{C^2} + (\partial + \partial \underset{001}{\mu})\partial \underset{11}{C_0} = \partial \underset{00}{D_1}. \quad (-4\cdot8)$$

Combining (±4·8), we obtain the fundamental equation as follows;

$$\begin{cases} \partial \underset{1111}{Y}\partial \underset{1}{C_0} + (\partial + \partial \underset{110}{\alpha})\partial \underset{00}{C^2} = \partial \underset{11}{D^1}, \\ (\partial + \partial \underset{001}{\mu})\partial \underset{11}{C_0} + \partial \underset{0000}{Z}\partial \underset{01}{C^2} = \partial \underset{00}{D_1}, \end{cases}$$

or

$$\left.\begin{aligned} &\underset{1}{D^{a}_{\kappa}}\underset{11}{Y^{\kappa\lambda}}\underset{1}{D^{b}_{\lambda}}\underset{1}{E_{b}} + (\underset{0}{K^{a}_{q}} + \underset{1100}{D^{a}_{\kappa}}\underset{100}{\alpha^{\kappa}_{\lambda}}\underset{0}{R^{\lambda}_{q}})\underset{0}{i^{q}} = \underset{1}{D^{a}_{\kappa}}\underset{1}{I^{\kappa}}, \\ &(-\underset{0}{K^{b}_{p}} + \underset{0010}{R^{\lambda}_{q}}\underset{010}{\mu^{\kappa}_{\lambda}}\underset{1}{D^{b}_{\kappa}})\underset{1}{E_{b}} + \underset{0}{R^{\lambda}_{p}}\underset{00}{Z_{\lambda\kappa}}\underset{0}{R^{\kappa}_{q}}\underset{0}{i^{q}} = \underset{0}{R^{\lambda}_{p}}\underset{0}{e_{\lambda}}. \end{aligned}\right\} \quad (4\cdot9)$$

Putting

$$\underset{1}{Y^{ab}_{11}} = \underset{1}{D^{a}_{\kappa}}\underset{11}{Y^{\kappa\lambda}}\underset{1}{D^{b}_{\lambda}}, \quad \underset{0\,00}{Z_{pq}} = \underset{0}{R^{\lambda}_{q}}\underset{00}{Z_{\lambda\kappa}}\underset{0}{R^{\kappa}_{p}},$$

$$\underset{100}{\alpha^{a}_{q}} = \underset{1100}{D^{a}_{\kappa}}\underset{100}{\alpha^{\kappa}_{\lambda}}\underset{0}{R^{\lambda}_{q}} \text{ and } \underset{010}{\mu^{b}_{p}} = \underset{00}{R^{\lambda}_{p}}\underset{010}{\mu^{\kappa}_{\lambda}}\underset{1}{D^{b}_{\kappa}},$$

(4·9) becomes

$$\begin{pmatrix} \underset{1}{Y^{ab}_{11}} & \underset{0}{K^{a}_{p}} + \underset{100}{\alpha^{a}_{p}} \\ -\underset{0}{K^{b}_{q}} + \underset{010}{\mu^{b}_{q}} & \underset{0\,00}{Z_{qp}} \end{pmatrix} \begin{pmatrix} \underset{1}{E_{b}} \\ \underset{0}{i^{p}} \end{pmatrix} = \begin{pmatrix} \underset{1}{I^{a}} \\ \underset{0}{e_{q}} \end{pmatrix}. \quad (4\cdot10)$$

Comparing this equation with (3·12), we find that the interaction between $\underset{1}{C_0}(\underset{1}{X})$ and $\underset{0}{C^2}(\underset{0}{X})$ is described, in this case, by means not only of the incidence matrix $\underset{0}{K^{a}_{q}}$, but also of $\underset{10}{\alpha}$ and $\underset{01}{\mu}$, which represent the mutual couplings of branches between $\underset{1}{X}$ and $\underset{0}{X}$.

In the case when $\underset{1}{Y^{ab}_{11}}$ is very simple, it is convenient to solve (4·10) by partitioning, be-

ginning with the inversion of Y_1^{ab}. This procedure is "diakoptics". In another case when Z_{000}^{pq} is very simple, it is convenient to begin with the inversion of Z_{000}^{pq} in solving (4·10) by partitioning. This procedure is "codiakoptics".

If many pairs of branches have mutual couplings, not only Y_1^{ab} and Z_{000}^{pq} but also $K+\alpha_{10}$ and $-K+\mu_{01}$, in our coefficient matrix of (4·10), have very complicated forms. Therefore, we shall have to proceed practically in a way similar to the ordinary partitioning of a matrix. On the contrary, if no mutual couplings exist between the branches of X_1 and the branches of X_0, then we have

$$\alpha_{10}=0,\quad \mu_{01}=0,$$

and

$$Y_1^{ab}=y_1^{ab},\quad Z_{000}^{pq}=z_{000}^{pq}.$$

Therefore, in such a case, (4·10) is reduced to the (3·12) of subsection 3·2.

Diakoptics and codiakoptics are feasible even when some mutual couplings exist between X_1 and X_0 provided they are simple.

4·3. Example: Transistor circuit. Let us analyze a transister circuit (Fig. 8). The T-equivalent circuit of a transister is shown in

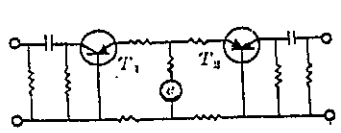


FIG. 8

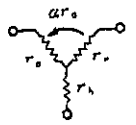


FIG. 9

Fig. 9. Therefore, we only need to analyze the equivalent circuit shown in Fig. 10. The

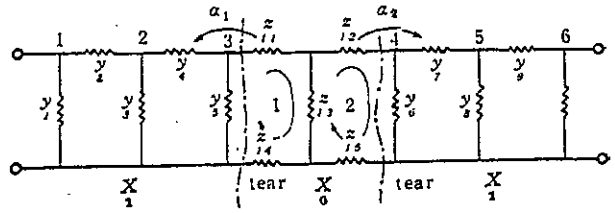


FIG. 10

branch characteristics of this equivalent circuit are shown by the mixed expression of Ohm's law as

$$\begin{pmatrix} Y_1 & \alpha_{10} \\ \mu_{01} & Z_0 \end{pmatrix} =$$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	y_1														
2		y_2													
3			y_3					0			α_1				
4				y_4											
5					y_5										
6						y_6									
7							y_7				α_2				
8		0						y_8							
9									y_9						
10										y_{10}					
11											z_{11}				
12												z_{12}			
13					0								z_{13}		
14														z_{14}	
15															z_{15}

where α_1 and α_2 are the α's (current amplification factors) of Transistor 1 and Transistor 2. (We can see that the mixed expression is very convenient for transistor and vacuum-tube circuits.) The fundamental equation can directly be written

$$
\begin{array}{c|ccc|ccc|cc|c}
 & 1 & 2 & 3 & 4 & 5 & 9 & 1 & 2 & \\
1 & y_1+y_2 & -y_2 & & & & & & & E_1 \\
2 & -y_2 & y_2+y_3+y_4 & -y_4 & & 0 & & \alpha_1 & & E_2 \\
3 & & -y_4 & y_4+y_5 & & & & 1-\alpha_1 & & E_3 \\
\hline
4 & & & & y_6+y_7 & -y_7 & & & 1-\alpha_2 & E_4 \\
5 & & 0 & & -y_7 & y_7+y_8+y_9 & -y_9 & & \alpha_2 & E_5 \\
6 & & & & & -y_9 & y_9+y_{10} & & & E_6 \\
\hline
1 & & -1 & & & & & z_{11}+z_{13}+z_{14} & -z_{13} & i^1 \\
2 & & & & & -1 & & -z_{13} & z_{12}+z_{13}+z_{15} & i^2
\end{array}
=
\begin{array}{c}
I^1 \\ I^2 \\ I^3 \\ I^4 \\ I^5 \\ I^6 \\ e_1 \\ e_2
\end{array}
\quad (4\cdot11)
$$

We solve (4·11) by "diakoptics", i. e. by partitioning the matrix beginning with the inversion of Y_1. The interaction terms between X_1 and X_0, i.e. $K_{11}+\alpha_{10}$ and $-K_{01}+\mu$, are so simple that we need little more than to invert two 3×3-matrices and one 2×2-matrix.

5. Procedures of Diakoptics and Codiakoptics

In this section, let us show the practical procedures of diakoptics and codiakoptics based on the topological foundations studied in the previous sections. That is, we solve the fundamental equation by partitioning of the coefficient matrix. These procedures will be shown to include the ordinary procedures of diakoptics and codiakoptics so far adopted by a number of investigators. For the sake of simplicity, we discuss only the case in which there are no mutual couplings between X_1 and X_0, but the generalization to more general cases will be carried out along the lines of subsection 4.

5·1. **Diakoptics.** In diakoptics, we solve the following equations

$$
\begin{cases}
\partial y_{111}\partial C_{100}+\partial\partial C^2_{00}=\partial D^1_{11}, \\
\partial\partial C_{011}+\partial z_{000}\partial C^2_{00}=\partial D_{10},
\end{cases}
$$

or the matrix equation

$$
\begin{pmatrix}
y^{ab}_{11} & K^{a}_{1\beta} \\
-K^{b}_{1\beta} & z_{\alpha\beta}
\end{pmatrix}
\begin{pmatrix}
E_b \\ i_\beta
\end{pmatrix}
=
\begin{pmatrix}
I^a \\ e_\alpha
\end{pmatrix}
\qquad (5\cdot1)
$$

by partitioning, beginning with the inversion of y^{ab}_{11}. We proceed as follows:

Solving the upper half of (5·1) with regards to E_b, we obtain

$$
E_b = z_{ab}I^a - z_{ab}K^{a\beta}i^0_\beta, \qquad (5\cdot2)
$$

where $z_{ba}=(y^{ab}_{11})^{-1}$ is regarded as the solution of the problem concerning the subnetworks. Calculating z_{ba}, as well as $z_{ba}I^a$ which is the node voltage of node b of the torn subnetworks, corresponds to solving the subnetworks.

Substituting (5·2) in the lower half of (5·1), we have

$$
z'_{\alpha\beta}i^\beta_0 = e_\alpha + K^{b}_{1\alpha}z_{ba}I^a \qquad (5\cdot3)
$$

where

$$
z'_{\alpha\beta}=z_{\alpha\beta}+K^{b}_{1\alpha}z_{ba}K^{a}_{1\beta} \qquad (5\cdot4)
$$

is the mesh-impedance matrix of the intersection network. Hence to obtain $z'_{\alpha\beta}$ is to compose the intersection network.

Solving (5·3), we obtain the mesh currents i^β_0 in X_0

$$
i^\beta_0 = y'^{\beta\alpha}(e_\alpha + K^{b}_{1\alpha}z_{ba}I^a), \qquad (5\cdot5)
$$

where

$$
y'^{\beta\alpha}=(z'_{\alpha\beta})^{-1}.
$$

Therefore, the intersection network is solved by this procedure.

The solution E_b of node b in X_1 is obtained by substituting (5·5) in (5·2)

$$
E_b = z_{ba}(I^a - K^{a}_{1\beta}i^\beta_0).
$$

The latter term, calculated from $i^{\overset{r}{0}}$ of (5·5), is the supplemented term by intersection network.

The branch currents and voltages are easily obtained from $(i^{\overset{r}{0}}, E_{\underset{1}{b}})$,

$$\left.\begin{aligned} i^{\overset{\kappa}{0}} &= R^{\overset{\kappa}{0}}_{\overset{r}{0}} i^{\overset{r}{0}}, \quad E_{\underset{0}{\lambda}} = z_{\underset{0}{\lambda}\underset{0}{\kappa}} i^{\overset{\kappa}{0}}, \\ i^{\overset{\kappa}{1}} &= y^{\overset{\kappa}{1}\underset{1}{\lambda}} E_{\underset{1}{\lambda}}, \quad E_{\underset{1}{\lambda}} = D^{\overset{a}{\lambda}}_{\underset{1}{\lambda}} E_{\underset{1}{a}}. \end{aligned}\right\} \qquad (5\cdot6)$$

We thus see that the diakoptical procedures coincide with our method of partitioning the fundamental equation.

These steps can be summarized in Diagram 1.

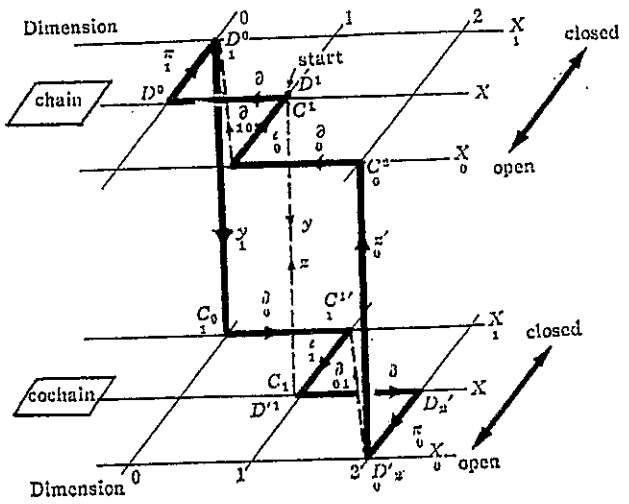


Diagram 1. Diakoptics

When the impressed quantities $D^0(I^a)$ and $D_0(e_q)$ are not yet explicitly given, we have only to calculate $z_{\underset{111}{b\,a}}$ and $y'^{\overset{r}{0}\overset{s}{0}}_{0}$ (these are called the factorized imverse matrices [2]). Thereafter, if these impressed quantities are given, we can immediately calculate the response quantities using $z_{\underset{111}{b\,a}}$ and $y'^{\overset{r}{0}\overset{s}{0}}_{0}$.

5·2. Codiakoptics. In codiakoptics, we solve the same equation (5·1)

$$\begin{pmatrix} y^{\overset{a}{1}\overset{b}{1}}_{\underset{1}{}} & K^{\overset{a}{1}}_{\underset{0}{r}} \\ -K^{\overset{b}{1}}_{\underset{0}{s}} & z_{\underset{0}{q}\underset{0}{r}} \end{pmatrix} \begin{pmatrix} E_{\underset{1}{b}} \\ i^{\overset{r}{0}}_{\underset{0}{}} \end{pmatrix} = \begin{pmatrix} I^{\overset{a}{1}} \\ e_{\underset{0}{q}} \end{pmatrix},$$

by partitioning the coefficient matrix, but beginning with the inversion of $z_{\underset{0}{q}\underset{0}{r}}$ in this case, we proceed as follows:

Solving the lower half of (5·1) with regards to $i^{\overset{r}{0}}$, we obtain

$$i^{\overset{r}{0}} = y^{\overset{r}{0}\overset{q}{0}}_{\underset{0}{}} e_{\underset{0}{q}} + y^{\overset{r}{0}\overset{q}{0}}_{\underset{0}{}} K^{\overset{b}{1}}_{\underset{0}{q}} E_{\underset{1}{b}}, \qquad (5\cdot7)$$

where $y^{\overset{r}{0}\overset{s}{0}}_{\underset{0}{}} = (z_{\underset{0}{q}\underset{0}{r}})^{-1}$. Calculating $y^{\overset{r}{0}\overset{q}{0}}_{\underset{0}{}}$, as well as $y^{\overset{r}{0}\overset{q}{0}}_{\underset{0}{}} e_{\underset{0}{q}}$, is to solve the subnetworks of codiakoptics.

Substituting (5·7) in the upper half of (5·1) we have

$$y'^{\overset{a}{1}\overset{b}{1}}_{\underset{1}{}} E_{\underset{1}{b}} = I^{\overset{a}{1}} - K^{\overset{a}{1}}_{\underset{0}{r}} y^{\overset{r}{0}\overset{q}{0}}_{\underset{0}{}} e_{\underset{0}{q}} \qquad (5\cdot8)$$

where

$$y'^{\overset{a}{1}\overset{b}{1}}_{\underset{1}{}} = y^{\overset{a}{1}\overset{b}{1}}_{\underset{1}{}} + K^{\overset{a}{1}}_{\underset{0}{r}} y^{\overset{r}{0}\overset{q}{0}}_{\underset{0}{}} K^{\overset{b}{1}}_{\underset{0}{q}}. \qquad (5\cdot9)$$

From (5·8), the node voltages $E_{\underset{1}{b}}$ in $X_{\underset{1}{}}$ are obtained as

$$E_{\underset{1}{b}} = z'_{\underset{11}{b\,a}} (I^{\overset{a}{1}} - K^{\overset{a}{1}}_{\underset{0}{r}} y^{\overset{r}{0}\overset{q}{0}}_{\underset{0}{}} e_{\underset{0}{q}}), \qquad (5\cdot10)$$

where

$$z'_{\underset{11}{b\,a}} = (y'^{\overset{a}{1}\overset{b}{1}}_{\underset{1}{}})^{-1}. \qquad (5\cdot11)$$

By equation (5·9) the intersection network is constructed, and (5·10) is the solution of the network.

Using (5·10), the mesh currents $i^{\overset{r}{0}}$'s are obtained

$$i^{\overset{r}{0}} = y^{\overset{r}{0}\overset{q}{0}}_{\underset{0}{}} (e_{\underset{0}{q}} + K^{\overset{b}{1}}_{\underset{0}{q}} E_{\underset{1}{b}}),$$

the latter term having been supplemented by the intersection network.

The branch currents and voltages are obtained by (5·6). Therefore, our method includes also the codiakoptical procedures.

These practical steps can be summarized in Diagram 2. It must be noted that Diagram 2 is the same as the previous diagram (Diagram 1) of diakoptics. The two are distinguished from each other only by their different starts. This shows that diakoptics and codiakoptics are essentially the same.

When the impressed quantities are not explicitly given, we have only to calculate $y^{\overset{r}{0}\overset{s}{0}}_{\underset{0}{}}$

and $z'_{\substack{ab\\11}}$. If these quantities are given, we can immediately calculate the response quantities using $y^{\substack{eg\\00}}_{0}$ and $z'_{\substack{ab\\11}}$.

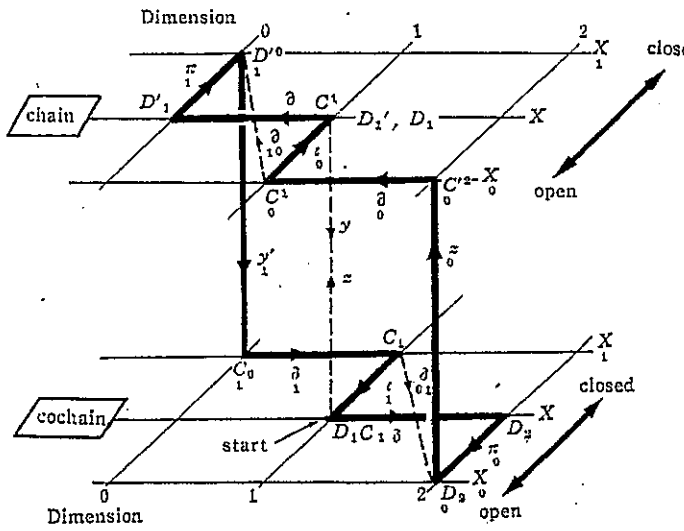


Diagram 2. Codiakoptics

6. Eigenvalue Analysis by means of Diakoptics

There has been some diakoptical analysis of eigenvalue problems by equivalent networks [2]. The methods employed so far being somewhat complicated, a simpler and more comprehensive formulation should be sought. This could be done using our fundamental equations with the diakoptical coordinates, as follows, by defining a normal form of the matrix of diakoptical eigenvalue problems. Two methods using the normal form will be shown.

6·1. Equation of eigenvalue problem by diakoptics. An eigenvalue problem is represented by

$$(A-\lambda I)\boldsymbol{x}=0, \qquad (6\cdot1)$$

where A is a symmetric square matrix, I the unit matrix, λ the eigenvalue to be determined, and $\boldsymbol{x}$ the eigenvector to be determined. In the case of the eigenvalue problem of a field,

$$H\varphi-\lambda\varphi=0, \qquad (6\cdot2)$$

where H is an hermitian operator and φ is the eigen-function, the general analytical problem can approximately be reduced to an algebraic problem such as (6·1) by approximating the differential operators by difference operators.

The problem (6·1) can easily be translated into an equivalent electrical network problem. In this representation, the eigenvector $\boldsymbol{x}$ is represented by the node-voltage vector $E=(E_a)$ (E_a is the voltage of a node a), and the matrix $A-\lambda I$ plays the rôle of the node-admittance matrix.

Since our node-admittance matrix is $A-\lambda I$, the network contains n branches having negative resistances whose admittances are the indeterminate eigenvalue $-\lambda$, where n is the order of the matrix, i. e. the number of the independent nodes. Every independent node is connected to the earth point by one of the branches whose admittances are $-\lambda$; for an example see Fig. 11.

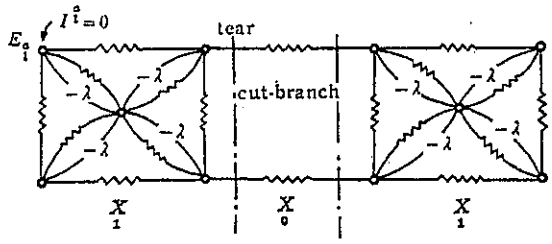
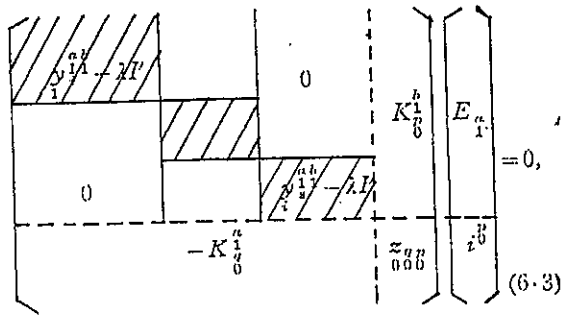


Fig. 11

Our eigenvalue problem is to obtain the eigenvalue λ, such that there appears a non-zero vector E_a without any impressed node currents, i. e. we have to solve

$$(A-\lambda I)E=0.$$

This can be done by diakoptics as follows. First, we dissect the network. In the present problem, let us dissect it in such a manner that the cut-network $\underset{0}{X}$ does not contain any nodes. Then the impedances of the cut-branches do not depend on λ. The subnetwork $\underset{1}{X}$ (which may consist of several disconnected parts) contains all the nodes. On using the diakoptical coordinates which consist of the node voltages $E_{\substack{a\\1}}(\underset{1}{a}=1,2,\ldots,n)$ and the mesh currents $i^{\substack{p\\0}}_{0}(\underset{0}{p}=1,2,\ldots,\underset{0}{k})$, the fundamental equation of diakoptics takes the following form



(6·3)

where each of the minors in the left upper half $y_1^{ab}-\lambda I'$ represents the node-admittance matrix of the i-th subnetwork torn apart, and z_{gg} is the mesh-impedance matrix of X_0 (represented by a diagonal matrix in our case in which each independent loop consists of one, and only one, cut-branch). $K_{\dot{p}0}^{b}$ is the incidence matrix between the node b of X_1 and the mesh (cut-branch) p of X_0, consisting of simple integers 0, ± 1. Thus we obtain the diakoptical equation of the eigenvalue problem. But we should note that equation (6·3) has not directly the type appearing in the ordinary eigenvalue problem, i. e. the coefficient matrix is not of the form $A-\lambda I$, but $A-\lambda I'$, where I' is not the unit matrix although it has a similar form;

$$I'=\begin{pmatrix} 1 & & & 0 & 0 \\ & 1 & & & \\ & & \ddots & & \\ & 0 & & 1 & \\ \hdashline 0 & & & 0 & \\ & \ddots & & & \\ & & 0 & & 0 \end{pmatrix}.$$

6·2. Normal form of the matrix of a diakoptical eigenvalue problem. In the previous subsection, we have obtained the equation of the eigenvalue problem of diakoptics,

$$\begin{pmatrix} y_1-\lambda I & K \\ \hdashline -K^T & z_0 \end{pmatrix}\begin{pmatrix} E_1 \\ \hdashline i_0 \end{pmatrix}=0. \qquad (6\cdot3)$$

But we see that the form of equation (6·3), or equivalently

$$\begin{pmatrix} y_1 & K \\ \hdashline -K^T & z_0 \end{pmatrix}\begin{pmatrix} E_1 \\ \hdashline i_0 \end{pmatrix}=\begin{pmatrix} \lambda & & & 0 \\ & \lambda & & \\ & & \ddots & \\ & & & \lambda \\ \hdashline 0 & & & 0 \end{pmatrix}\begin{pmatrix} E_1 \\ \hdashline i_0 \end{pmatrix}, \qquad (6\cdot4)$$

is not invariant under basis transformations of the diakoptics coordinates.

Since I' is not invariant, i. e. $T^{-1}I'T\neq I'$ or

$$T^{-1}\begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ \hdashline 0 & & & 0 \end{pmatrix}T\neq\begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ \hdashline 0 & & & 0 \end{pmatrix},$$

for an arbitrary non-singular matrix T, the Jacobian method cannot be effective for solving (6·4). But if transformation is made only in the subspace composed of E_1, without changing the i_0-part, the matrical equation (6·4) is kept form-invariant.

Theorem 2. *By a unitary transformation of the E_1-part, the diakoptical equation of the eigenvalue problem can be transformed into the following form*

$$\begin{pmatrix} \lambda_1'-\lambda & & & K' \\ & \lambda_2'-\lambda & & \\ & & \ddots & \\ & & \lambda_n'-\lambda & \\ \hdashline -K'^T & & & z_0 \end{pmatrix}\begin{pmatrix} E_1' \\ \hdashline i_0 \end{pmatrix}=0 \qquad (6\cdot5)$$

where λ_i' is the i-th eigenvalue of the subnetwork X_1.

Proof: Let u_i be the i-th normalized eigenvector of the subnetwork, and

$$U=[u_1 u_2 \cdots\cdots u_n].$$

Then U is a unitary matrix, $U^{-1}=U^T$. Adopting these vectors u_i as the new basis vectors of the sub-space E_1, the matrix y_1^{ab} of the subnetwork is transformed into the diagonal form

$$U^{-1}y_1U=\begin{pmatrix} \lambda_1' & & & 0 \\ & \lambda_2' & & \\ & & \ddots & \\ 0 & & & \lambda_n' \end{pmatrix}.$$

Therefore, we have (6·5), where the eigenvectors of the subnetwork X_1 are chosen as the basis vectors of the E_1-part, and $K'=U^TK$.

We call (6·5) "the diakoptical normal form

of the equation of eigenvalue problems".

To solve the problem by diakoptics is to solve firstly the subnetwork and secondly the intersection network by utilizing the solution of the subnetwork. Therefore, we can start with the normal form (6·5) which can be obtained from the solution of the subnetwork. Several methods will be found for so handling (6·5). Here we shall propose only two of them, one of which is that which was propounded by Kron [2].

6·3. Solution by direct method (Kron's method). We shall first show Kron's method [2] in a slightly revised form represented in terms of our theory. Let us consider the diakoptical normal form of the problem represented by

$$\begin{pmatrix} \lambda'_1-\lambda & & & K' \\ & \ddots & & \\ & & \lambda'_n-\lambda & \\ & -K'^T & & z \end{pmatrix} \begin{pmatrix} \underset{1}{E'} \\ \underset{0}{i} \end{pmatrix} = 0,$$

where $\lambda'_i\,(i=1,\ldots n)$ is the i-th eigenvalue of the torn subnetwork. Solving this equation by partitioning and eliminating $\underset{1}{E}$, we obtain

$$z'(\lambda)\underset{0}{i} = \left(\underset{0}{z} + K'^T \begin{pmatrix} \lambda'_1-\lambda & & \\ & \ddots & \\ & & \lambda'_n-\lambda \end{pmatrix}^{-1} K'\right)\underset{0}{i} = 0. \tag{6·6}$$

Therefore, the non-zero vector $\underset{0}{i}$ can appear when, and only when, the determinant of the matrix $z'(\lambda)$ vanishes

$$|z'| = \left| z + K'^T \begin{pmatrix} \frac{1}{\lambda'_1-\lambda} & & \\ & \ddots & \\ & & \frac{1}{\lambda'_n-\lambda} \end{pmatrix} K' \right| = 0. \tag{6·7}$$

Solving this characteristic equation, we obtain the eigenvalues λ_t of the entire system. Using the eigenvectors $\underset{0}{i_j}$ corresponding to the eigenvalue $\lambda=\lambda_j$ of (6·6), we have the eigenvector $\underset{1}{E_j}$ from the upper half of (6·5)

$$\begin{pmatrix} \lambda'_1-\lambda_j & & & \\ & \lambda'_2-\lambda_j & & 0 \\ & & \ddots & \\ & 0 & & \lambda'_n-\lambda_j \end{pmatrix} \begin{pmatrix} \\ \underset{1}{E'_j} \\ \\ \end{pmatrix} = -K'\underset{0}{i_j}, \tag{6·8}$$

in reference to the basis $(u_1, u_2, \ldots u_n)$ consisting of the set of the eigenvectors of the subnetwork. The i-th eigenvector $\underset{1}{E_j}$ will be obtained as

$$\underset{1}{E_j} = U\underset{1}{E'_j}. \tag{6·9}$$

A special treatment will be needed for such a degenerate case that there appear some non-zero $\underset{1}{E}$'s in spite of $\underset{0}{i}=0$, no current appearing in the cut-branches. In such a case, the above consideration cannot be applied, and the equation (6·5) reduces to

$$\begin{pmatrix} \lambda'_1-\lambda & & & \\ & \lambda'_2-\lambda & & \\ & & \ddots & \\ & & & \lambda'_n-\lambda \\ & & -K'^T & \end{pmatrix} \begin{pmatrix} \\ \\ \underset{1}{E'} \\ \\ \end{pmatrix} = 0. \tag{6·10}$$

Non-zero $\underset{1}{E'}$ can appear only when λ coincides with one of the λ'_i's, i. e., an eigenvalue coincides with one of those of the subnetwork. Generally, such may be the case when more than two eigenvalues of the subnetwork coincide. The eigenvector $\underset{1}{E'}$ satisfying (6·10) will easily be found to be a linear combination of the eigenvectors of the subnetwork corresponding to the multiple eigenvalue. When the subnetwork has some multiple eigenvalues, we must solve the problem after eliminating the degrees of freedom corresponding to these degenerated eigenvectors.

The above method essentially coincides with Kron's (cf. [2]), except that the treatment of the degenerate case is revised.

6·4. Solution by iterative method.[1] A method

1) The original idea for this method was obtained during our discussions with Dr. M. Iri on this theme. This method has apparently some similarity with the escalator method [27].

can now be developed for obtaining all the eigenvalues exactly by k_0 iterations (k_0 is the number of the cut-branches). This is, physically speaking, to connect a cut-branch to the subnetwork and solve the system so obtained, in one iterative process. Then, regarding this new system as the new subnetwork, we iterate this procedure till all the cut-branches are connected.

First, solving the eigenvalue problem of the subnetwork, we may start with the normal form,

$$\begin{pmatrix} \lambda_1-\lambda & & & & K' & E' \\ & \lambda_2-\lambda & & & & \\ & & \ddots & & & \\ & & & \lambda_n-\lambda & & \\ & & -K'^T & & z & i \end{pmatrix}_{0} \begin{matrix} \} n \\ \} k \end{matrix} = 0, \tag{6·11}$$

where λ_i is the i-th eigenvalue of the subnetwork, $K'=U^TK$ and U is the unitary matrix composed of the eigenvectors of the subnetwork. Before we start on the iterative procedures, it may be convenient to eliminate those degrees of freedom which have the same eigenvalues as those of the subnetwork by the method described in the previous subsection (the degenerate case). Of course, this is not a necessary procedure, but it is convenient.

Let us connect a cut-branch, say 1, to the subnetwork. Then the equation of the connected system has the normal form

$$\begin{pmatrix} \lambda_1-\lambda & & & & K'_1 & E'' \\ & \lambda_2-\lambda & & & & \\ & & \ddots & & & \\ & & & \lambda_n-\lambda & & \\ & & -K'^T_1 & & z_1 & i^0 \end{pmatrix}_{0,1} \begin{matrix} \} n \\ \} 1 \end{matrix} = 0, \tag{6·11}$$

where K'_1, is the first column of K' and z_1 is the impedance of the first cut-branch. Expanding the determinant of the coefficient matrix of (6·11) with respect to the last column, we have for the characteristic equation,

$$z_1(\lambda_1-\lambda)(\lambda_2-\lambda)\cdots(\lambda_n-\lambda)+\sum_{i=1} k'^2_{1i}(\lambda_1-\lambda)\cdots(\lambda_{i-1}-\lambda)(\lambda_{i+1}-\lambda)\cdots(\lambda_n-\lambda)=0, \tag{6·12}$$

where k'_{1i} is the i-th component of K'_1. If some components of K'_1, say k'_{1j}, are zero, we obtain from (6·12) a solution

$$\lambda=\lambda_j.$$

This is a degenerate case, and its eigenvector is the same as u_j, the j-th eigenvector of the subnetwork. If a multiple root λ_j exists in the subnetwork, this is also the eigenvalue of the new system. Hence, we can immediately find the eigenvalue and the eigenvector in a degenerate case without any calculation. Afterwards we divide (6·12) by $\prod_{i=1}^{n}(\lambda_i-\lambda)$ and obtain

$$f(\lambda)=z_1+\frac{k'^2_{11}}{\lambda_1-\lambda}+\frac{k'^2_{12}}{\lambda_2-\lambda}+\cdots+\frac{k'^2_{1n}}{\lambda_n-\lambda}=0. \tag{6·13}$$

Since $f(\lambda)$ has poles at $\lambda=\lambda_i\,(i=1,\ldots,n)$ and $z_1>0$, $k'^2_{1i}>0$, we see that the i-th zero-point of $f(\lambda)$, λ_i lies between λ_i and λ_{i+1},

$$\lambda_1\leqq\lambda_1\leqq\lambda_2\leqq\lambda_2\leqq\cdots\leqq\lambda_n\leqq\lambda_n. \tag{6·14}$$

Since $f(\lambda)$ and $f'(\lambda)=\frac{df(\lambda)}{d\lambda}$ have simple forms such as (6·13) and

$$f'(\lambda)=\sum_{i=1}\frac{k'^2_{1i}}{(\lambda_i-\lambda)^2}, \tag{6·14}$$

and the positions of the roots are bounded by (6·14), the roots will easily be obtained by Newton's method. Thus we can calculate the eigenvalues λ_i's $(i=1,2,\ldots n)$ of the subnetwork to which the first cut-branch is added. The eigenvectors u_i's of the newly obtained subnetwork are obtained from (6·11) by,

$$\begin{pmatrix} \frac{1}{\lambda_1-\lambda_i} & & \\ & \ddots & \\ & & \frac{1}{\lambda_n-\lambda_i} \end{pmatrix}_{0,1} \begin{pmatrix} \\ K'_1 \\ \end{pmatrix}. \tag{6·15}$$

Thus we have the eigenvalues λ_i and the eigenvectors u_i, hence a unitary matrix $U = [u_1, u_2, \ldots, u_n]$ of the new subnetwork. Then we iterate the above procedure by connecting another cut-branch, say 2, to the subnetwork where the branch 1 is already connected. when k iterations have been carried out, we obtain the eigenvalues and the eigenvectors of the original system exactly.

We have the following theorem with respect to this method.

Theorem 3. *The i-th eigenvalue λ_i of the j-th iterative procedure lies between λ_{i-1} and λ_{i-1},*

$$\lambda_{i-1} \leqq \lambda_i \leqq \lambda_{i+1}.$$

6·5. Example. Determine the vertical vibration of the mechanical system shown in Fig. 12 where four masses have weights 1;

$$m=1,$$

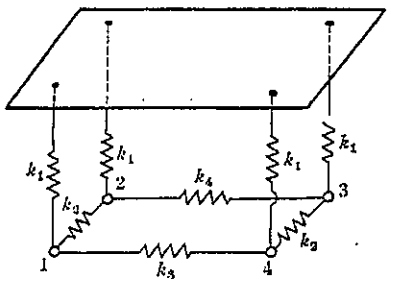


FIG. 12

and the stiffnesses of the springs are respectively

$$k_1=2,\ k_2=1,\ k_3=\frac{1}{3},\ k_4=\frac{1}{2}.$$

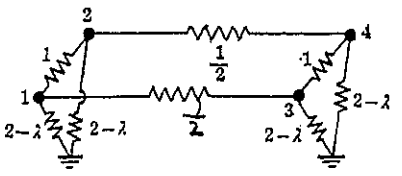


FIG. 13

This system can be represented equivalently by an electrical network shown in Fig. 13, where the node voltages E_a's $(a=1, 2, 3, 4)$ represent the amplitude of the vibration of the masses, and $\lambda=\omega^2$ (ω is the angular frequency to be determined). The admittances of the branches are shown in the figure (Fig. 13). The ordinary method shows the eigenvalue problem such that

$$\begin{pmatrix} 3\frac{1}{3}-\lambda & -1 & -\frac{1}{3} & 0 \\ -1 & 3\frac{1}{2}-\lambda & 0 & -\frac{1}{2} \\ -\frac{1}{3} & 0 & 3\frac{1}{3}-\lambda & -1 \\ 0 & -\frac{1}{2} & -1 & 3\frac{1}{2}-\lambda \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{pmatrix} = 0. \tag{6·16}$$

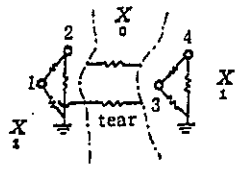


FIG. 14

Tearing the network (see Fig. 14), the solution of each separated subdivision can easily be obtained. Since the two parts of the torn subnetwork are the same, we may solve only one of them, i. e. the following equation

$$\begin{pmatrix} 3-\lambda' & -1 \\ -1 & 3-\lambda' \end{pmatrix} \begin{pmatrix} E'_1 \\ E'_2 \end{pmatrix} = 0.$$

From this we have the solution of the subnetwork, whose eigenvalues are

$$\lambda'_1=2,\quad \lambda'_2=4,\quad \lambda'_3=2,\quad \lambda'_4=4,$$

and whose eigenvectors stand as the columns in a unitary matrix

$$U=\frac{1}{\sqrt{2}}\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 1 & & \\ 1 & -1 & & \\ & & 1 & 1 \\ & & 1 & -1 \end{pmatrix} \end{array}. \tag{6·17}$$

The incidence matrix $K_{\beta_0}^{\alpha_1}$ is

$$K_{\beta_0}^{\alpha_1}=\begin{array}{c} \alpha_1 \backslash \beta_0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} & 1 \\ 1 & \\ & -1 \\ -1 & \end{pmatrix} \end{array}.$$

There are two double roots, $\lambda'=2$ and $\lambda'=4$.

We find from (6·10) that a linear combination of the eigenvectors corresponding to $\lambda'=2$, $(u_1+u_3)/\sqrt{2}$ is the eigenvector of the entire system of which the eigenvalue is $\lambda=2$. Another eigenvector of the subnetwork corresponding to $\lambda=2$ and orthogonal to this vector is $(u_1-u_3)/\sqrt{2}$. Of course, this is not the eigenvector of the entire system. In a similar way we find that $(u_2+u_4)/\sqrt{2}$ is the eigenvector of the entire system corresponding to $\lambda=4$. Another eigenvector of the subnetwork corresponding to $\lambda'=4$ is $(u_2-u_4)/\sqrt{2}$. Hence taking

$$\frac{1}{\sqrt{2}}(u_1+u_3),\ \frac{1}{\sqrt{2}}(u_2+u_4),$$
$$\frac{1}{\sqrt{2}}(u_1-u_3),\ \frac{1}{\sqrt{2}}(u_2-u_4),$$

as the new basis vectors (they are orthogonal to each other), the diakoptical normal form of the equation can be rewritten

$$\left(\begin{array}{cc|cc:c} 2-\lambda & & 0 & & 0 \\ & 4-\lambda & & & \\ \hline & & 2-\lambda & & 1\ \ 1 \\ 0 & & & 4-\lambda & -1\ \ 1 \\ \hdashline & & -1 & 1 & 2 \\ 0 & & & & \\ & & -1 & -1 & \ \ 3 \end{array}\right)\left(\begin{array}{c} E_1'' \\ E_2'' \\ \hline E_3'' \\ E_4'' \\ \hdashline i_0^1 \\ i_0^2 \end{array}\right)=0.$$

The first two rows represent the trivial solutions we have obtained in the above, i. e. $\lambda=2$, $\lambda=4$ and

$$u_1=\frac{1}{2}[1\ 1\ 1\ 1],\quad u_2=\frac{1}{2}[1\ -1\ 1\ -1].$$

The other parts yield the matric equation of the normal form

$$\left(\begin{array}{cc:cc} 2-\lambda & & 1 & 1 \\ & 4-\lambda & -1 & 1 \\ \hdashline -1 & 1 & 2 & \\ -1 & -1 & & 3 \end{array}\right)\left(\begin{array}{c} E_3'' \\ E_4'' \\ \hdashline i_0^1 \\ i_0^2 \end{array}\right)=0. \quad (6\cdot19)$$

The solution of this equation can easily be obtained by either of the direct or the iterative method. They yield

$$\lambda_3=2.819540,\ \lambda_4=4.847127,$$

and

$$u_3=[.539537\ .457056\ -.539537\ -.457056],$$
$$u_4=[.457056\ -.539537\ -.457056\ .539537].$$

7. Topological Structures of $\overset{1}{X}$ and $\overset{0}{X}$

As we solve the network problems of X and $\overset{1}{X}$ in diakoptics or codiakoptics, we must clarify the topological structures thereof. Since the homology groups of $\overset{1}{X}$ have such unusual characteristics as do not appear in X, we shall first investigate them. Then, adding some meshes or nodes to X, we shall obtain the equivalent network of $\overset{1}{X}$. Using it, we shall study how many unknown variables are needed in this method.

7·1. Homology groups of $\overset{1}{X}$. Since $\overset{1}{X}$ is closed, a 1-cell (branch) of $\overset{1}{X}$ is incident to one, and only one, 0-cell (node) of $\overset{1}{X}$ with positive sign and to one, and only one, with negative sign. Hence the character of the incidence relations between 0-cells and 1-cells is the same in $\overset{1}{X}$ as in an ordinary network complex. Therefore, we can conclude (see [24]) that there are no integral torsion coefficients in the dimension 0

$$\overset{1}{T}{}_1^0=0,$$

and the 0-th Betti number $\overset{1}{R}{}^0$ is equal to the number of (connected) components of $\overset{1}{X}$. Therefore, the 0-dimensional integral homology group $\overset{1}{H}{}_1^0(X)$ is a free group with $\overset{1}{R}{}^0$ generating elements. But the 1-dimensional homology group $\overset{1}{H}{}_1^1(X)$ of $\overset{1}{X}$ is quite different from the $H_1^1(X)$ of X, not only in that it has a torsion part but also that it may have even a Betti part[1], i. e. $\overset{1}{R}{}^1\neq 0$, while the 1-dimensional homology group $\overset{1}{H}{}_1^1(X)$ of $\overset{1}{X}$ always vanishes. As there are no 3-dimensional elements in $\overset{1}{X}$

1) In X, every 1-dimensional cycle can be made to bound a 2-cell or a 2-chain [24]. Consequently the 1-dimensional torsion group, as well as the Betti group, can be reduced to nil. But there may appear the 1-dimensional torsion and Betti groups, if the boundaries of meshes (2-cells) added to some loops (1-cycles) of X do not constitute a basis of the group of integral 1-cycles. (see [25].)

$H^2_J(X)$ is a free group of rank R^2. Thus we have

Theorem 4. *The structures of the homology groups of X are represented as follows:*

$$\left.\begin{aligned} H^0_J(X) &= B^0_J(X), \\ H^1_J(X) &= B^1_J(X) + T^1_J(X), \\ H^2_J(X) &= B^2_J(X). \end{aligned}\right\} \qquad (7\cdot1)$$

Hence it should be noted that a 1-cycle is not necessarily a 1-boundary in X.

7·2. Homology groups of X. Next, a 1-cell of X is incident at most to one 0-cell of X with positive sign and at most to one with negative sign. Therefore, there are no 0-dimensional integral torsion coefficients in X,

$$T^0_J(X) = 0.$$

Since X is not closed[1], there is at least one node of X in every connected component of X. Therefore, every node of X, by adding an appropriate node of X, bounds in X, that is, every node of X bounds in X, hence the 0-dimensional Betti number R^0 of X is 0. Thus the 0-dimensional homology group $H^0_J(X)$ vanishes

$$H^0_J(X) = 0.$$

Since X is open, every 2-cell incident to a 1-cell of X is in X. Let us assume such an equation as $\partial C^2 = tC^1$, where t is an integer and $t \geqq 2$. Then, choosing an appropriate chain C^1 of X, we can construct a cycle $C^1 + C^1$ and this cycle bounds in X since $H^1_J(X)$ vanishes, so we have $\partial C'^2 = C^1 + C^1$. Thus we have $\partial \pi C'^2 = C^1$, which contradicts the assumption $t \geqq 2$. Therefore the 1-dimensional torsion group of X vanishes, $H^1_J(X) = 0$. But the first Betti group of X does not vanish. Then the first homology group $H^1_J(X)$ of X is free group of rank R^1. The 2nd homology group $H^2_J(X)$ of X is also a free group of rank R^2, since there are no 3-dimensional elements. Thus we have

Theorem 5. *The structures of the homology groups of X are represented as follows:*

$$\left.\begin{aligned} H^0_J(X) &= 0, \\ H^1_J(X) &= B^1_J(X), \\ H^2_J(X) &= B^2_J(X). \end{aligned}\right\} \qquad (7\cdot2)$$

Hence, a 1-cocycle is not necessarily a 1-coboundary in X.

Since the 0-th Betti number of X is R_0, there are R_0 independent 0-cocycles. A physically realizable voltage configuration being determined by the coboundary of a 0-cochain, the node voltage 0-cochain $C_0 = E_a \sigma^a_0$ in X has R_0-fold indeterminacy. As a base of the group of 0-cocycles of X, we can choose the fundamental 0-cocycles of each component (a fundamental 0-cocycle is the sum of all nodes contained in one component). Therefore, this indeterminacy physically corresponds to the indeterminacy of the absolute node voltage of the reference point of each component of X (this can be determined by considering C_0 of X). By taking the coboundary of C_0, this indeterminacy vanishes.

Dually to the above, there are R^2 independent 2-cycles, so our $C^2 = \sum i^0 \sigma^2_0$ has R^2-fold indeterminancy, but it vanishes by taking the boundary of C^2_0.

7·3. Interrelations between various homology groups and cohomology groups. The structure of the homology groups of X are clarified above. We can know the structures of the cohomology groups of X from the 1st duality theorem (p. 117 of [4]), i. e.

$$B^r_J(X) \cong B^J_r(X) \text{ and } T^r_J(X) \cong T^J_{r+1}(X),$$

or equivalently

$$R^r = R_r \text{ and } t^r_k = t^k_{r+1} \quad (i = 0, 1;\ k = 1, 2, \ldots).$$

1) If X is closed X is simultaneously open and closed. This means that X is X itself (since $R^1 = 1$, i. e. X is a connected network). Such trivial tearing is excluded from consideration in this section.

We know the following exact sequences of homomorphisms[1] as the interrelations of various homology groups [22];

$$0\longrightarrow \underset{(R^2)}{H^2(X)} \xrightarrow{\bar{i}} \underset{(R^2)}{H^2(X)} \xrightarrow{\bar{\pi}} \underset{(R^2)}{H^2(X)} \xrightarrow{\bar{\partial}} \underset{(R^1, t^1_i,)}{H^1(X)}$$

(Betti numbers, torsion coefficients)

$$\xrightarrow{\bar{i}} 0 \xrightarrow{\bar{\pi}} \underset{(R^1)}{H^1(X)} \xrightarrow{\bar{\partial}} \underset{(R^0)}{H^0(X)} \xrightarrow{\bar{i}} \underset{(R^1=1)}{H^0(X)} \xrightarrow{\bar{\pi}} 0 \quad [2]$$

$H^1(X)$ $H^0(X)$

and those of various cohomology groups:

$$0\xrightarrow{\bar{i}} H_0(X) \xrightarrow{\bar{\pi}} H_0(X) \xrightarrow{\bar{\partial}} H_1(X) \xrightarrow{\bar{i}} 0$$

$H_0(X)$ $H_1(X)$

$$\xrightarrow{\bar{\pi}} H_1(X) \xrightarrow{\bar{\partial}} H_2(X) \xrightarrow{\bar{i}} H_2(X) \xrightarrow{\bar{\pi}} H_2(X)$$

$$\xrightarrow{\bar{\partial}} 0.$$

From this sequence, we obtain the relations

$$R^2_1 + R^1_1 = R^2_1 + R^2_0 \tag{7·3}$$

and

$$R^0_1 = R^1_0 + 1. \tag{7·4}$$

7·4. Addition of meshes and nodes. It is very inconvenient for practical analysis that subnetworks X_1 and X_0 have such unusual characteristics as mentioned above. These are given rise to at dissection only imaginarily (for there are neither meshes nor nodes, in reality, but only loops and cut-sets) and have no important practical meaning. Therefore, we can eliminate the first homology group $H^1_1(X_1)$ by adding some meshes (2-cells) to X_1, as we did in the first part of this paper to the ordinary network X. We can also eliminate the first homology group $H^1_1(X_0)$ by adding some nodes (0-cells) to X_0. However, let us first investigate the number of independent nodes m_1 of X_1 and the number of independent meshes k_0 of X_0.

Denoting the number of the r-dimensional elements of X_i by α^r_i, we obtain

$$m_1 = \alpha^0_1 - R^0_1, \tag{7·5}$$

and

$$k_0 = \alpha^2_0 - R^2_0. \tag{7·6}$$

Using the Euler-Poincaré relation for each subcomplex

$$\sum_r (-1)^r \alpha^r_i = \sum_r (-1)^r R^r_i$$

or

$$(\alpha^2_i - R^2_i) + (\alpha^0_i - R^0_i) = (\alpha^1_i - R^1_i), \tag{7·7}$$

we obtain

$$k_0 = \alpha^2_0 - R^2_0 = (\alpha^1_0 - R^1_0) - (\alpha^0_0 - R^0_0).$$

Considering (7·4), i. e. $R^0_1 = R^1_1 + 1$, and $R^0_0 = 0$, we obtain

$$k_0 = \alpha^1_0 - (\alpha^0_0 + R^0_1 - 1). \tag{7·8}$$

Since

$$\alpha^0_0 + R^1_0 = \alpha^0_0 + R^0_1 - 1$$

is the number of the independent 1-cocyles (cut-sets) in X_0, we define m_0 by

$$m_0 \overset{\text{def}}{=} \alpha^0_0 + R^0_1 - 1. \tag{7·9}$$

Then we have from (7·8)

$$k_0 + m_0 = \alpha^1_0. \tag{7·10}$$

Let us construct a new complex (network) $\bar{X}_0$ by adding to X_0, R^0_1 nodes which correspond to all connected components of X_1 one-to-one. See Fig. 15. Therefore the first homology group $H^1_1(\bar{X}_0)$ vanishes and m_0 can be regarded as the number of the independent nodes of $\bar{X}_0$. Practically, we had better use this $\bar{X}_0$ instead of X_0, and Kirchhoff's laws in $\bar{X}_0$ are expressed in the

---

1) A sequence of homomorphisms
$$\cdots\longrightarrow A_{r-1}\longrightarrow A_r\longrightarrow A_{r+1}\longrightarrow\cdots$$
is said to be exact if for each integer r, the image of
$$\tau_{r-1}: A_{r-1}\longrightarrow A_r$$
coincides with the kernel of
$$\tau_r: A_r\longrightarrow A_{r+1}.$$

2) $\bar{\tau}: H(X)\longrightarrow H(Y)$ means the homomorphism induced by
$$\tau: Z(X)\longrightarrow Z(Y)\ (\tau: F(X)\longrightarrow F(Y)).$$

same way as in the ordinary network.

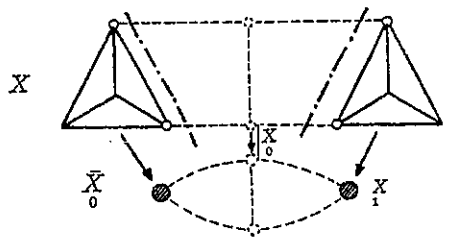


FIG. 15

In a similar way as above, we have

$$m_1 = \alpha^1_1 - (\alpha^2_1 + R^2_0 - R^2). \qquad (7\cdot11)$$

Since

$$\alpha^2_1 + R^1_1 - R^2_1 = \alpha^2_1 + R^2_0 - R^2$$

is the number of the indepentent 1-cycles in X_1, we define k_1 by

$$k_1 = \alpha^2_1 + R^2_0 - R^2. \qquad (7\cdot12)$$

Then we have from (7·11)

$$k_1 + m_1 = \alpha^1_1. \qquad (7\cdot13)$$

Let us construct a new complex $\bar{X}_1$ by adding $R^2_0 - R^2$ meshes to X_1 and changing some meshes in such a way that every 1-cycle may become the boundary of a 2-cell or 2-cells (meshes). Then the first homology group $H_1(X)$ vanishes, and k_1 can be regarded as the number of the independent meshes of X_1. Practically, it would be better to use the $\bar{X}_1$ instead of X_1, and, of course, Kirchhoff's laws are expressed in the same way as in the ordinary network.

In F-VII [8], we shall use these $\bar{X}_0$ and $\bar{X}_1$ to calculate the number of independent meshes and nodes.

7·5. Number of independent unknown variables. The number of the components of the coordinates (the number of the independent unknown quantities or the rank of the matrix to be inverted) in diakoptics or codiakoptics is the sum of those of X_1 and X_0. Kron said in his paper (*Part II: Orthogonal Networks* of [2]), "When the original uncut system has N equations, and the system is subdivided into n subdivisions, a set of k unknowns appears at the points of cut (k in number). Hence it is necessary to solve for $N+k$ unknowns in $n+1$ independent groups, instead of solving for only N simultaneous unknowns. The solution of these additional k unknowns is the price the engineer has to pay for whatever advantage accrues from the piecemeal solution of N unknowns".[1] But this is not valid for our more general cases. Using (7·10) and (7·13), the number of variables in diakoptics or codiakoptics is shown to be

$$m_1 + k_0 = (\alpha^0_1 - R^0_1) + (\alpha^2_0 - R^2_0) = \alpha^1_0 - m_0 - k_1. \qquad (7\cdot14)$$

As m_0 (the number of independent cut-sets in X_0) and k_1 (the number of independent loops in X_1) have negative sign in (7·14), whereas $\alpha^1_0 + \alpha^1_1 = \alpha^1 = n$ is a constant (the total number of the branches), we can sometimes make $m_1 + k_0$ (the number of variables) decrease by dissection less than m and k. It would be better to dissect X_1, roughly speaking, in such a way that X_1 contains the parts of many meshes (large mesh-density) compared with the number of nodes and X_0 contains the parts of many nodes (large node-density) compared with the number of meshes. We shall show two extreme examples.

Example in Fig. 16. *In the network shown in Fig. 16, X_1 consists of two completely connected subnetworks*[2], *each having s nodes, and X_0 is a*

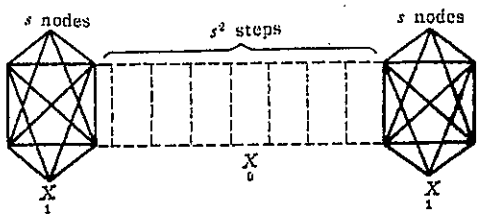


FIG. 16

1) This statement is obviously valid in such cases as Kron's in which each subdivision of X_1 has the same grounded point in X_1, and X_0 (cut-branches in Kron's terminology) includes no 1-dimensional elements (nodes). By our definition, however, diakoptics has been extended to a more general method.

2) A completely connected network is such a network that each pair of nodes is connected by one, and only one, branch.

ladder type network, having s^2 steps. If s is very large, the number of the node coordinates or the mesh coordinates is about $2s^2$, to each case, whereas the number in diakoptics is about s^2, reducing to one half of the former.

Example in Fig. 17. *In the network shown in Fig. 17, X_1 is a completely connected network with $2s$ nodes and X_0 consists of s ladder-type networks, each having $2s$ steps. If s is very large, the number of the unknown quantities associated with the mesh-type method (i. e. of the mesh coordinates) or the node-type method is about $4s^2$, whereas the number in codiakoptics is about $2s^2$, reducing to one half of the former.*

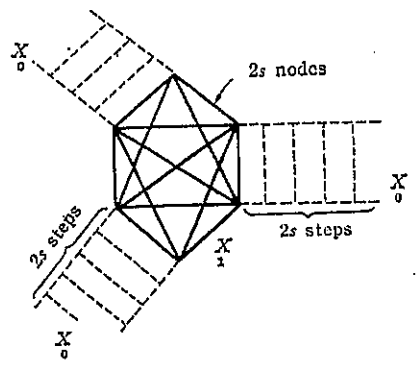


FIG. 17

REFERENCES

[1] S. Amari, On the Topological Foundations of Diakoptics and Codiakoptics. *RAAG Research Notes, Third Series*, No. 21 (December 1959).

[2] G. Kron, Diakoptics—The Piecewise Solution of Large-Scale Systems, Chapters I~XX. *The Electrical Journal*, London, 1957~1959.

[3] R. Onodera, Diakoptics and Codiakoptics of Electrical Networks. *RAAG Memoirs*, 2, F-II (1958), 369~388.

[4] S. Lefschetz, *Algebraic Topology*. American Mathematical Society Colloquium Publications, 27, New York, 1942.

[5] S. Amari, On the Information Contained in the Graph of a Network and Practical Methods of Dissection in Diakoptics. *RAAG Research Notes, Third Series*, No. 25 (March 1960).

[6] S. Amari, Diakoptical Analysis of Eigenvalue Problems. *RAAG Research Notes, Third Series*, No. 27 (May 1960).

[7] S. Amari, *On the Topological and Information-Theoretical Foundations of Diakoptics and Codiakoptics.* A Master's Thesis submitted to the Division of Research in Mathematical and Physical Sciences of the Graduate School of the University of Tokyo, March 1960.

[8] S. Amari, Information-Theoretical Foundations of Diakoptics and Codiakoptics. *RAAG Memeirs*, 3, F-VII (1962), 351~371.

[9] G. Kron, Diakoptics—The Science of Tearing, Tensors and Topological Models. *RAAG Memoirs*, 2, F-I (1958), 343~368.

[10] G. Kron, Numerical Example for Interconnecting Piece-Wise Solutions of Elastic Structures. *RAAG Memoirs*, 2, F-III (1958), 389~403.

[11] J. P. Roth, The Validity of Kron's Method of Tearing. *Proceedings of the National Academy of Sciences, U. S. A*, 41., 8 (1955), 599~600.

[12] R. R. Sabroff and T. J. Higgins, A Critical Study of Kron's Method of "Tearing". *The Matrix and Tensor Quarterly*, 7, 4; 8, 1, 2, 3, 4; 9, 1, 2; June 1957~December 1958.

[13] R. R. Sabroff, New Concepts and Generalizations of Kron's Method of Tearing. *The Matrix and Tensor Quarterly*, 10, 1, 2, 3, 4; September 1959~June 1960.

[14] R. Onodera, A New Approach to Kron's Methods of Analysing Large Systems. *Institution for Electrical Enginees, Monograph*, 403 *E*, October 1960.

[15] K. Kondo, On "Generalized Diakoptics". *RAAG Memoirs*, 2, F-V (1958), 409~422.

[16] M. Iri and T. Sunaga, A Comment of the Efficacy of Diakoptical Analysis from an Information-Theoretical Viewpoint. *RAAG Memoirs*, 2, F-IV (1958), 404~408.

[17] J. P. Roth, An Application of Algebraic Topology. Kron's Method of Tearing. *Quarterly of Applied Mathematics*, 17, 1 (1959), 1~24.

[18] G. Kron, *Tensor Analysis of Networks.* John Wiley and Sons, New York, 1939.

[19] W. A. Blackwell and H. H. Kesvan, Simplification of the Analysis of Large Systems by Linear Graph Techniques. *The Matrix and Tensor Quarterly*, 11, 1 (1960), 11~22.

[20] J. P. Char, Solution of Orthogonal Network. *The Matrix and Tensor Quarterly*, 8, 4 (1958), 102~105.

[21] J. P. Char, Solution of Orthogonal Networks—Admittance Method. *The Matrix and Tensor Quarterly*, 10, 1 (1959), 1~5,

[22] S. Eilenberg and N. Steenrod, *Foundatiqns of Algebraic Topology*. Princeton University Press, Princeton, 1952.

[23] Papers included in Division. A, Linear Geometry of Topology of Networks. *RAAG Memoirs*, 1, 2, and 3, 1955, 1958 and 1962.

[24] K. Kondo and M. Iri, On the Theory of Trees, Cotrees, Multi-Trees and Multi-Cotrees. *RAAG Memoirs*, 2, A-VII (1958), 42~83.

[25] M. Iri, Supplements to the Theory of Trees, Cotrees, Multitrees and Multicotrees. *RAAG Memoirs*, 3, A-X (1962), 4~22.

[26] Y. Mizoo, M. Iri and K. Kondo, On the Torsion and Linkage Characteristics and the Duality of Electric, Magnetic and Dielectric Networks. *RAAG Memoirs*, 2, A-VII (1958), 84~117.

[27] J. Morris, *The Escalator Method in Engineering Vibration Problems*. Chapman & Hall, London, 1947.